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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

STATISTICAL METHODS FOR THE ANALYSIS OF RE-ENTRY VEHICLE RADAR DATA PART I

L. A. GARDNER, JR.

Group 22

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ABSTRACT

This report treats the theoretical problem of joint estimation of the wake and hard body cross sections (the parameters of the Rice distribution) from either incoherent or coherent radar data.

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1. INTRODUCTION

During re-entry into the atmosphere of a hypersonic ballistic missile the character of a radar return changes in several respects from that of the return observed when the vehicle is in the exoatmosphere. Many of these differences (in particular, the often rapid and random variations in amplitude and phase) are believed to be due to the presence of a turbulent wake acting as a statistical scattering medium. It is generally assumed that there exists a random distribution of turbulent eddies which are large enough in number, within one resolution volume of the viewing radar, to permit application of the Central Limit Theorem. In other words, the resultant scattered field is due to the superposition of a large number of components with independent phases. This then implies that the resultant quadrature, or the Cartesian components of the scattered field, form a pair of independent Gaussian stochastic process. Thus, the amplitudes of the received signal are Rayleigh distributed, i.e. the received power has an exponential distribution.

Often the resultant signal at the radar is not that due to the turbulent wake alone, but is a superposition of this component with the coherent return from a hard body scatterer. Such a superposition may be observed in the range cell containing the hard body which, because of the size of the resolution volume, also contains some of the turbulent wake. In the case of an extended return, the same situation may obtain even when the observations are not made on the range cell containing the body. This is due to range, and perhaps angle, sidelobes of the body return.

In this report the statistics of the signal received when the hard body and wake returns are combined are investigated. Four computationally feasible methods for estimating the component powers,arising separately from the turbulent wake and the hard body, are developed and compared. The first two methods utilize amplitude data only, and thus are suitable for general use with the data obtained by most radars. The third and fourth methods utilize phase returns as well as amplitudes, and thus presuppose a coherent radar. An intermediate step in the third method also yields

an estimate of Doppler frequency.

Specifically, then, the raw data for a coherent radar are pulse-to-pulse amplitudes and phases within a series of range cells. In accordance with the preceding discussion, the model hypothesized for the amplitude r_t and phase p_t recorded on the t^{th} pulse is, when written in Cartesian coordinates,

$$\begin{aligned} r_t \cos p_t &= x_t = \rho \cos \varphi_t + \sigma w_{1t} \\ r_t \sin p_t &= y_t = \rho \sin \varphi_t + \sigma w_{2t} \end{aligned} \qquad (t = 1, 2, \ldots, n). \tag{1.1}$$

(The notation does not indicate the fixed range cell.) $\{w_{1t}\}$ and $\{w_{2t}\}$ denote independent Gaussian processes, each of which is presumed white with mean 0 and variance 1. For the present, a linear phase model is postulated:

$$\varphi_t = \omega t + \varphi. \tag{1.2}$$

The immediate problem is joint estimation of the wake and hard body cross section parameters

$$\theta_1 = 2\sigma^2 \qquad \theta_2 = \rho^2 \tag{1.3}$$

without knowledge of the phase parameters. We are chiefly interested in rapidly computable estimates, i.e. those having storage requirements which do not depend on the sample size n.

All evaluations will be made on the basis of results of large sample distribution theory. In practice, n cannot be taken too large, because the assumption of a constant hard body amplitude would most certainly be unrealistic. Nonetheless,

asymptotic comparisons have qualitative comparative value when considering fixed sample size applications. In addition, the labor required to obtain higher order approximations to mean square errors is not worth the effort when the validity of the model is still open to question. In particular, there is empirical indication that the regressions might be superpositions of sinusoids, or that the wake processes are not white. These questions are deferred until a later report.†

Since there may be an additional acceleration term in (1. 2), it is customary (for incoherent radars, necessary) to focus attention on the amplitudes $r_t = \sqrt{x_t^2 + y_t^2}$, which are distributed the same for every t and independently of the phase model. The common variate, say r, is said to be Rice distributed when $\rho \neq 0$ (see p. 238 in Wax [7]), and Rayleigh distributed in the special case $\rho = 0$. In statistical terminology, r/σ has a noncentral chi distribution with 2 degrees of freedom (written χ'_2) depending only on the parameter ρ. When ρ is 0, the adjective "noncentral" is dropped and the prime deleted from the chi. This distribution obtains for any two mean value functions whose sum of squares is ρ^2 (see e.g. Kendall and Stuart [4], p. 227). There is a certain notational ease inherent in the statistical terminology which makes it worth while using in the course of analysis.

2. SUMMARY OF THE METHODS

Throughout we will let

$$a_k = \frac{1}{n}\sum_{t=1}^{n} r_t^k \qquad \alpha_k = \mathcal{E} r^k \tag{2. 1}$$

respectively denote the sample and population amplitude moments of integer order k. The dependence of the former on n is not shown and must be remembered.

Method I is based on the fact that

† The results which we derive in Sec. 7, and specialize to the analysis of the third and fourth methods, apply to more general models.

$$\frac{\alpha_1}{\alpha_2^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{2} \frac{e^{-\frac{\lambda}{2}}}{(1+\lambda)^{\frac{1}{2}}} \left[(1+\lambda) I_0 \left(\frac{\lambda}{2}\right) + \lambda I_1 \left(\frac{\lambda}{2}\right) \right] = F(\lambda)$$

where I_ν is the modified Bessel function of the first kind of order ν, and

$$\lambda = \theta_2 / \theta_1 \tag{2.2}$$

is the signal-to-noise ratio (or, more appropriately, the hard body - to - wake ratio). The function F increases monotonically from $F(0) = \sqrt{\pi}/2$ to $F(\infty) = 1$. The ratio (2.2) can therefore be estimated by

$$\lambda^* = \begin{cases} F^{-1}(a_1/a_2^{\frac{1}{2}}) & \text{if} \quad a_1/a_2^{\frac{1}{2}} \geq \sqrt{\pi}/2 \\ 0 & \text{otherwise .} \end{cases} \tag{2.3}$$

Then, since

$$\alpha_2 = \text{expected received power} = \theta_1 + \theta_2 \text{ or } \theta_1(1+\lambda),$$

the quantities

$$\boxed{\text{I.} \qquad \theta_1^* = \frac{a_2}{1+\lambda^*} \qquad \theta_2^* = a_2 - \theta_1^*} \tag{2.4}$$

are taken as estimates of the parameters (1.3). Figure 1 is included to provide a means for roughly estimating λ from the observed ratio $a_1/a_2^{\frac{1}{2}}$.†

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† The estimates (2.4) have been considered by D. Batman of MIT Lincoln Laboratory.

We point out here that the (or any) Maximum Likelihood estimates, based on amplitude data, satisfy

$$\hat{\theta}_1 + \hat{\theta}_2 = a_2 .$$

We will invariably use this as one of our estimator equations.

Method II is based on the fact that the moments of r^2/θ_1 are just Laguerre polynomials in $-\lambda$. As a consequence

$$\theta_2 = (2\alpha_2^2 - \alpha_4)^{\frac{1}{2}} .$$

Using the first two sample power moments, then, we immediately have

$$\boxed{\text{II.} \quad \theta_1^* = a_2 - \theta_2^* \qquad \theta_2^* = \begin{cases} (2a_2^2 - a_4)^{\frac{1}{2}} & \text{if } 2a_2^2 \geq a_4 \\ 0 & \text{otherwise} \end{cases}} \tag{2.5}$$

as consistent estimates of (θ_1, θ_2). It should be noted that both pairs (θ_1^*, θ_2^*) are necessarily nonnegative.

For either method of estimation, the distribution of

$$\sqrt{n}\,(\theta_1^* - \theta_1) \qquad \sqrt{n}\,(\theta_2^* - \theta_2)$$

tends to a zero mean bivariate normal distribution as $n \to \infty$. In each case, the covariance matrix is of the form

$$\underline{\Sigma} = \theta_1^2 \, \underline{Q}(\lambda) \qquad (\lambda > 0).$$

The joint asymptotic efficiency of Method II relative to Method I is customarily measured by the ratio of determinants (called generalized variances)

$$d(\lambda) = \frac{|Q_I(\lambda)|}{|Q_{II}(\lambda)|}, \tag{2.6}$$

where the meaning of affixes is clear. In spite of the fact that Method II is computationally simpler than Method I, since the latter requires table look-up and interpolation for F^{-1}, the function (2.6) exceeds unity throughout some neighborhood of $\lambda = 0$ (below some point just short of .4). Both methods become asymptotically efficient in the limit of indefinitely large λ, and Method II also does as $\lambda \to 0$. This is not too interesting since the generalized variance of the ML estimates tend to infinity quadratically as $\lambda \to 0$ and linearly as $\lambda \to \infty$.

The poor statistical behavior of (2.4) and (2.5) for small values of λ can be eliminated when we use the phase data, and assume the model (1.2) holds with some arbitrary frequency and initial phase.

The time series in (1.1) are clearly nonstationary. They are, however, "asymptotically stationary." The reason for this terminology is that

$$A_h = \frac{1}{n}\sum_{t=1}^{n-|h|} x_t x_{t+|h|} \qquad B_h = \frac{1}{n}\sum_{t=1}^{n-|h|} y_t y_{t+|h|} \qquad (|h| \leq n-1)$$

converge as $n \to \infty$, both with probability one and in mean square, to certain limits. It is true for their sum that

$$S_h = A_h + B_h \to \theta_1 \delta_{h,0} + \theta_2 \cos h\omega \equiv \gamma_h \qquad (-\infty < \omega < \infty,\ \lambda \geq 0) \tag{2.7}$$

for every fixed h, where δ_{ij} is Kronecker's delta. The Fourier transform of this sequence ($h = 0, \pm 1, \pm 2, \ldots$) is constant at all angular frequencies save at ω, where it has a jump of magnitude θ_2 over and above the level θ_1. This fact suggests basing estimation schemes on empirical (windowed) power spectra. Such, however, entail calculations of a number of covariances which in principle increases without bound

with sample size. For the present, we will not be concerned with spectral techniques.

On the other hand, we can still consistently estimate (θ_1, θ_2) by various simple combinations of the first few covariances. These methods can't attain full efficiency, but there is an improvement in variances over Methods I and II which increases without bound as λ gets smaller. Method III is one such procedure which first estimates

$$\xi = \cos\omega \tag{2.8}$$

by

$$\xi^* = \frac{S_2 + \sqrt{S_2^2 + 8S_1^2}}{4S_1} . \tag{2.9}$$

This is derived from the fact that $\{x_t\}$ and $\{y_t\}$ obey the same second order stochastic difference equation driven by independent second order moving averages with common variance $\theta_1(1+2\xi^2)$. (2.9) is obtained as the (statistically equivalent) unambiguous point minimizing

$$\frac{1}{1+2\xi^2} \sum_{t=3}^{n} [(x_t - 2\xi x_{t-1} + x_{t-1})^2 + (y_t - 2\xi y_{t-1} + y_t)^2] . \tag{2.10}$$

This particular criterion ignores the first two correlations of the driving processes. By taking them into account one obtains a greatly improved estimate of ξ, at the expense of computational loads depending on n. On the other hand, if the variance is also ignored, i.e. the multiplier in (2.10) is deleted, one obtains the estimate known in numerical analysis as Prony's method. It is inconsistent for any finite λ.

From (2.7) we have the identity

$$\gamma_2 - 2\xi\gamma_1 + \gamma_0 = \theta_1$$

between the parameters. We can therefore estimate (θ_1, θ_2) by

$$\boxed{\text{III.} \quad \theta_1^* = S_2 - 2\xi^* S_1 + S_0 \quad \theta_2^* = S_0 - \theta_1^*} \tag{2.11}$$

We still get consistency if we replace S_h by A_h in ξ^* and by $2B_h$ in θ_1^*. At first sight it might be expected that this alteration would lead to better estimates, because ξ^* would be independent of the remaining random variables appearing in θ_1^*. This is not the case. In fact, the variances are uniformly larger than those of the estimates as written.†

Method III by itself cannot distinguish between the interval $(0, \pi)$ and $(\pi, 2\pi)$ of angular frequency. However, by combining (2.9) with the first cross-covariance we can consistently estimate ω (mod 2π). In general, we have for any fixed integer h

$$2C_h = \frac{2}{n} \sum_{t=1}^{n-h} x_t y_{t+h} \rightarrow \begin{cases} \sin 2\varphi\theta_2 \cos h\omega & \text{if } \omega = 0, \pm\pi, \pm 2\pi, \ldots \\ \\ \theta_2 \sin h\omega & \text{otherwise .} \end{cases} \tag{2.12}$$

Thus

$$\omega^* = \begin{cases} \cos^{-1}\xi^* \text{ in } (0, \pi) & \text{if } C_1 \geq 0 \\ \\ \cos^{-1}\xi^* + \pi & \text{if } C_1 < 0 \end{cases}$$

converges strongly to $\omega \neq k\pi$. The sampling properties of this estimate become poor when ω is near a multiple of π, because the variance of $\cos^{-1}\xi^*$ is $1/\sin^2\omega$ times

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†The altered version of θ_1^* was previously suggested. Cf. $Q^2(\xi)$ on p. 8 of 22L-7529 and P_{11} of (8.3) below.

that of ξ^*, and the latter does not reach 0 at these points.

If ω is not a multiple of π, we can combine (2.7) and the second line of (2.12) to give another pair of consistent estimates:

$$\boxed{\text{IV.} \quad \theta_1^* = S_0 - \theta_2^* \qquad \theta_2^* = \sqrt{S_1^2 + (2C_1)^2}} \tag{2.13}$$

This method is preferable to Method III when λ is sufficiently small (depending on ξ), but the relative asymptotic efficiency approaches 0 as λ increases. If ω is a multiple of π, then

$$\theta_2^* \to \theta_2 \sqrt{1+\sin^2 2\varphi}$$

as $n \to \infty$. Thus, unless the unknown initial phase in (1.2) is also some multiple of π, the estimate is inconsistent.

3. AMPLITUDE MOMENTS

Here we derive certain formulae which will be needed to compute the large sample distributions of Methods I and II.† The reader who is already familiar with the properties of the Rice distribution and its moments can skip to Sec. 4, and refer back to this section when it is indicated.

It is convenient to work with the parameters (θ_1, λ) when doing the analysis, and to simplify the writing we will usually drop the subscript 1. Thus,

$$\theta = 2\sigma^2 \qquad \lambda = \rho^2/2\sigma^2 . \tag{3.1}$$

Throughout we adopt the functional definitions used in [1]. Any symbol [x.y.z.] refers to the equation so numbered in this handbook.

† Although most of these appear in existing literature, we include the following derivations in order to have a permanent record of verified results.

Let x_t and y_t be any independent normal random variables with common variance σ^2, and respective means μ_{1t} and μ_{2t} for which

$$\mu_{1t}^2 + \mu_{2t}^2 = \rho^2$$

for all integers t. The joint probability element of (x_t, y_t) is

$$\frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[(x-\mu_{1t})^2 + (y-\mu_{2t})^2]} dxdy.$$

Changing to polar coordinates, the probability that r_t fall in $(r, r+dr)$ and p_t in $(p, p+dp)$ is

$$dF = \frac{e^{-\lambda}}{\pi\theta} re^{-\frac{r^2}{\theta}} e^{u_{1t}\cos p + u_{2t}\sin p} drdp$$

$$u_{it} = \frac{2\mu_{it}}{\theta} r \qquad (i = 1, 2).$$

Writing $u_1 = R\cos\alpha$ and $u_2 = R\sin\alpha$, periodicity gives

$$\int_0^{2\pi} e^{u_1\cos p + u_2 \sin p} dp = \int_{-\alpha}^{2\pi-\alpha} e^{R\cos\varphi} d\varphi$$

[9.6.16] $$= \int_0^{2\pi} e^{R\cos\varphi} d\varphi = 2\pi\, I_0(R).$$

The series expansion for the modified Bessel function of the first kind of general order ν is

[9.6.10] $$I_\nu(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+\nu}}{n!\,\Gamma(n+\nu+1)} \,. \tag{3.2}$$

Thus, after integrating the phase out of dF we obtain

$$f(r;\theta,\lambda) = \frac{2e^{-\lambda}}{\theta}\, r e^{-\frac{r^2}{\theta}}\, I_0\left(2\sqrt{\frac{\lambda}{\theta}}\, r\right) \qquad (r \geq 0) \tag{3.3}$$

for the time-independent probability density function of $r_t = \sqrt{x_t^2 + y_t^2}$.

If we substitute I_0's infinite series we find (3.3) expressed as a "mixture" of central chi densities with Poisson weights:

$$f(r;\theta,\lambda) = \sum_{n=0}^{\infty} c_n(\lambda)\, f_{2n+2}(r;\theta) \tag{3.4}$$

wherein

$$c_n(\lambda) = e^{-\lambda}\,\frac{\lambda^n}{n!}$$

$$f_\nu(r;\theta) = \frac{2}{\Gamma(\frac{\nu}{2})\,\theta^{\frac{\nu}{2}}}\, r^{\nu-1} e^{-\frac{r^2}{\theta}} \quad .$$

The latter is the probability density function of $\sqrt{\theta/2}\cdot\chi_2$. It might be remarked in passing that the form (3.4) generalizes to many dimensions: The density of $r = \sqrt{x_1^2 + \cdots + x_q^2}$, when the x's are independent and x_i is normal with mean μ_i and variance σ^2, is given by (3.4) with the subscript $2n+2$ replaced by $2n+q$ and ρ^2 redefined as $\mu_1^2 + \cdots + \mu_q^2$.

The k^{th} moment of χ_ν is

$$\mathcal{E}\chi_\nu^k = \frac{1}{2^{\frac{\nu}{2}-1}\,\Gamma(\frac{\nu}{2})} \int_0^\infty r^{\nu+k-1} e^{-\frac{1}{2}r^2}\, dr = \frac{2^{\frac{k}{2}}}{\Gamma(\frac{\nu}{2})}\,\Gamma\left(\frac{\nu+k}{2}\right).$$

Hence, with the parameter dependence not shown,

$$\alpha_k = \int_0^\infty r^k f(r)dr = (\tfrac{1}{2}\theta)^{\frac{k}{2}} \sum_{n=0}^{\infty} c_n \, \lambda^{k}_{2n+2}$$

$$= \theta^{\frac{k}{2}} e^{-\lambda} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{k}{2}+1)}{(n!)^2} \lambda^n . \qquad (3.5)$$

For theoretical purposes it is best to express α_k in terms of Kummer functions

[13. 1. 2] $$M(a, b, z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+a)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(n+b)} \frac{z^n}{n!} .$$

Thus

$$\alpha_k = \theta^{\frac{k}{2}} \Gamma(\frac{k}{2}+1) e^{-\lambda} M(\frac{k}{2}+1, 1, \lambda)$$

[13. 1. 27] $$= \theta^{\frac{k}{2}} \Gamma(\frac{k}{2}+1) M(-\frac{k}{2}, 1, -\lambda). \qquad (3.6)$$

The latter form agrees with Rice's formula (p. 339 of Wax [7]), who uses the confluent hypergeometric notation ${}_1F_1$ in place of M.

The formula for the even moments reduces to a Laguerre polynomial. In fact,

$$\alpha_{2k} = \theta^k k!\, M(-k, 1, -\lambda) = \theta^k k!\, L_k(-\lambda)$$

[13. 6. 9] $$L_k(-\lambda) = \sum_{j=0}^{k} \binom{k}{j} \frac{\lambda^j}{j!} \qquad (3.7)$$

This can be derived directly from (3. 5) by first noting that

$$\frac{(n+k)!}{n!}\lambda^n = \left(\frac{d}{d\lambda}\right)^k \lambda^{n+k},$$

and then using Rodrigue's formula.

We examine the behavior of the amplitude moments for large λ. From (3.7) we immediately see that

$$\alpha_{2k} \simeq (\theta\lambda)^k$$

as $\lambda \to \infty$. For the odd moments we use the expansion

[13.5.1] $$M(-\frac{k}{2}, 1, -\lambda) = \frac{\lambda^{k-\frac{1}{2}}}{\Gamma(k+\frac{1}{2})}\left\{\sum_{n=0}^{N-1} [(\tfrac{1}{2}-k)_n]^2 \frac{\lambda^{-n}}{n!} + O(\lambda^{-N})\right\}$$

wherein

$$(\tfrac{1}{2}-k)_n = \frac{\Gamma(n+\frac{1}{2}-k)}{\Gamma(\frac{1}{2}-k)} \quad .$$

Since

[6.1.17] $$\frac{1}{\Gamma(k+\frac{1}{2})\,\Gamma(\frac{1}{2}-k)} = \frac{1}{\pi}\sin\pi(k+\tfrac{1}{2}) = \frac{(-1)^k}{\pi}$$

we find for (3.6), after replacing k by $2k-1$,

$$\alpha_{2k-1} \simeq (\theta\lambda)^{\frac{2k-1}{2}}\left\{1 + \frac{\Gamma^2(k+\frac{1}{2})}{\pi}\sum_{n=1}^{N-1}\frac{\Gamma^2(n+\frac{1}{2}-k)}{\pi\, n!}\lambda^{-n} + O(\lambda^{-N})\right\} \qquad (3.8)$$

as $\lambda \to \infty$. Consequently, the k^{th} moment of r/ρ tends to 1 as λ increases, which is obvious for θ tending to 0 with ρ fixed. In a sense made precise by Rice on p. 240 in Wax [7], the density of r looks normal about ρ with variance $\frac{1}{2}\theta$ (not θ) when ρ is large and θ is fixed.

We will need α_1 and α_3, made independent of θ after diversion by α_2 raised to the appropriate power. For computational purposes, it is best to write these as linear combinations of I_0 and I_1. The connection is made via

$$M(\tfrac{1}{2}, 1, \lambda) = e^{\frac{\lambda}{2}} I_0(\tfrac{\lambda}{2})$$

[13.6.3] $$M(\tfrac{3}{2}, 3, \lambda) = e^{\frac{\lambda}{2}} \frac{4}{\lambda} I_1(\tfrac{\lambda}{2})$$

and certain recurrence relations among 3 M's at a fixed argument. From the first formula in (3.6) we see that α_1 requires $M(\frac{3}{2}, 1, \lambda)$ and α_3 requires $M(\frac{5}{2}, 1, \lambda)$. If we multiply [13.4.2] with $a=3/2$, $b=2$ and $z=\lambda$ by λ, and add to it $-2(1+\lambda)$ times [13.4.4] with $a=3/2$, $b=1$ and $z=\lambda$, we obtain a formula for $M(\frac{3}{2}, 1, \lambda)$ in terms of the above two M's. The result is

$$\alpha_1 = \theta^{\frac{1}{2}} \frac{\sqrt{\pi}}{2} e^{-\frac{\lambda}{2}} [(1+\lambda) I_0(\tfrac{\lambda}{2}) + \lambda I_1(\tfrac{\lambda}{2})]. \tag{3.9}$$

We get $M(\frac{5}{2}, 1, \lambda)$ in terms of $M(\frac{1}{2}, 1, \lambda)$ and $M(\frac{3}{2}, 1, \lambda)$ directly from [13.4.1] with $a=3/2$, $b=1$ and $z=\lambda$. Thus,

$$\alpha_3 = \theta^{\frac{3}{2}} \frac{3\sqrt{\pi}}{4} e^{-\frac{\lambda}{2}} [(1+2\lambda+\tfrac{2}{3}\lambda^2) I_0(\tfrac{\lambda}{2}) + \tfrac{2}{3}\lambda(2+\lambda) I_1(\tfrac{\lambda}{2})]; \tag{3.10}$$

or, α_3 can be written in terms of I_0 and α_1.

4. ANALYSIS OF METHODS I AND II

From (3.7) the mean power is

$$\alpha_2 = \theta(1+\lambda)$$

and thus by (3.9)

$$F(\lambda) = \frac{\alpha_1}{\alpha_2^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{2} \frac{e^{-\frac{\lambda}{2}}}{(1+\lambda)^{\frac{1}{2}}} [(1+\lambda) I_0(\tfrac{\lambda}{2}) + \lambda I_1(\tfrac{\lambda}{2})] \tag{4.1}$$

Using (3.2) and (3.8) we see that

$$F(0) = \sqrt{\pi}/2 = .8862\ 2693$$

$$F(\infty) = 1.$$

Since

[9.6.28] $\quad \frac{d}{dz} I_0(z) = I_1(z) \qquad \frac{d}{dz} I_1(z) = I_0(z) - \frac{1}{z} I_1(z)$

we find for the derivative of F

$$F'(\lambda) = \frac{\sqrt{\pi}}{4} \frac{e^{-\frac{\lambda}{2}}}{(1+\lambda)^{3/2}} I_1(\tfrac{\lambda}{2}) > 0 \tag{4.2}$$

Thus, F is a strictly increasing function of its argument, and the estimate (2.3) of λ is indeed well-defined.

To compute the large sample distribution of (2.4) we need that of (a_1, a_2), of which (θ_1^*, θ_2^*) are functions. In general, it is true that (Cramér [2], p. 364)

$$\sqrt{n}(a_{k_1} - \alpha_{k_1}),\ \sqrt{n}(a_{k_2} - \alpha_{k_2}), \ldots, \sqrt{n}(a_{k_p} - \alpha_{k_p})$$

tend to joint normality as $n \to \infty$ with zero mean vector and covariance matrix with ij^{th} entry

$$\alpha_{k_i + k_j} - \alpha_{k_i}\alpha_{k_j} = n\, \mathcal{E}(a_{k_i} - \alpha_{k_i})(a_{k_j} - \alpha_{k_j}) \tag{4.3}$$

$$(i, j = 1, 2, \ldots, p)$$

In particular, for $\sqrt{n}(a_1 - \alpha_1)$ and $\sqrt{n}(a_2 - \alpha_2)$ we have the covariance matrix

$$\underline{A} = \begin{bmatrix} \alpha_2 - \alpha_1^2 & \alpha_3 - \alpha_1\alpha_2 \\ \alpha_3 - \alpha_1\alpha_2 & \alpha_4 - \alpha_2^2 \end{bmatrix} .$$

We express the entries in terms of $F(\lambda)$ and (see (3.10))

$$G(\lambda) = \frac{\alpha_3}{\alpha_2^{3/2}}$$

$$= \frac{3\sqrt{\pi}}{4} \frac{e^{-\frac{\lambda}{2}}}{(1+\lambda)^{3/2}} \left[(1 + 2\lambda + \frac{2}{3}\lambda^2) I_0(\frac{\lambda}{2}) + \frac{2}{3}\lambda(2+\lambda) I_1(\frac{\lambda}{2}) \right] . \qquad (4.4)$$

We then have, using (3.7),

$$\alpha_2 - \alpha_1^2 = \theta(1+\lambda)(1 - F^2)$$

$$\alpha_3 - \alpha_1\alpha_2 = \theta^{3/2}(1+\lambda)^{3/2}(G-F)$$

$$\alpha_4 - \alpha_2^2 = \theta^2(1+2\lambda)$$

where the argument value is understood.

We apply the delta method (see Appendix A), and introduce the functions

$$g = g(x_1, x_2) = \begin{cases} \dfrac{x_2}{1 + F^{-1}(x_1/x_2^{\frac{1}{2}})} & \text{if } x_1/x_2^{\frac{1}{2}} \geq \sqrt{\pi}/2 \\ x_2 & \text{otherwise} \end{cases}$$

$$g_1 = g \qquad\qquad g_2 = x_2 - g .$$

In this notation the estimates (2.4) are

$$\theta_1^* = g_1(a_1, a_2) \qquad \theta_2^* = g_2(a_1, a_2)\,.$$

Under the assumption $\lambda \neq 0$, i.e. $\alpha_1/\alpha_2^{\frac{1}{2}} > \sqrt{\pi}/2$, the matrix of derivatives evaluated at the true parameter values is

$$\underline{D} = \left[\begin{matrix} \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_2}{\partial x_1} \\ \\ \dfrac{\partial g_1}{\partial x_2} & \dfrac{\partial g_2}{\partial x_2} \end{matrix}\right]_{\underline{x} = \underline{\alpha}} = \left[\begin{matrix} h_1 & -h_1 \\ \\ h_2 & 1-h_2 \end{matrix}\right]$$

wherein

$$h_1 = \left.\frac{\partial g}{\partial x_1}\right|_{\underline{x} = \underline{\alpha}} = -\theta^{\frac{1}{2}} \frac{1}{(1+\lambda)^{3/2} F'}$$

$$h_2 = \left.\frac{\partial g}{\partial x_2}\right|_{\underline{x} = \underline{\alpha}} = \frac{1}{1+\lambda} + \frac{F}{2(1+\lambda)^2 F'}\,.$$

The covariance matrix of the limiting normal distribution of $\sqrt{n}(\theta_1^* - \theta_1)$ and $\sqrt{n}(\theta_2^* - \theta_2)$ is

$$\underline{\Sigma} = \underline{D}' \, \underline{A} \, \underline{D} = \theta^2 \underline{Q} \tag{4.5}$$

where the symmetric matrix $\underline{Q} = [Q_{ij}]$ depends only on λ. We express its elements in terms of F, G and

$$H = (1+\lambda)F'\,. \tag{4.6}$$

After considerable algebraic manipulation we arrive at

$$\boxed{\begin{aligned}
&\text{I.}\\
&Q_{11} = \frac{1}{H^2}\left[1-F^2 - 2(G-F)(H+\tfrac{1}{2}F) + \frac{1+2\lambda}{(1+\lambda)^2}(H+\tfrac{1}{2}F)^2\right]\\
&Q_{12} = -Q_{11} - \frac{1}{H}\left[(1+\lambda)(G-F) - \frac{1+2\lambda}{1+\lambda}(H+\tfrac{1}{2}F)\right]\\
&Q_{22} = -Q_{11} - 2Q_{12} + 1+2\lambda
\end{aligned}} \qquad (4.7)$$

.

The determinant of $\underline{Q}$ is easy to compute because that of $\underline{D}$ is simply h_1. We have

$$|\underline{\Sigma}| = h_1^2 |\underline{A}| = h_1^2 \theta^3 \begin{vmatrix} (1+\lambda)(1-F^2) & (1+\lambda)^{3/2}(G-F) \\ (1+\lambda)^{3/2}(G-F) & 1+2\lambda \end{vmatrix}$$

$$= \frac{\theta^4}{(1+\lambda)H^2}\left[(1+\lambda)(1+2\lambda)(1-F^2) - (1+\lambda)^3 (G-F)^2\right] .$$

Consequently,

$$\boxed{\begin{aligned}
&\text{I.}\\
&|\underline{Q}| = \frac{1}{H^2}\left[(1+2\lambda)(1-F^2) - (1+\lambda)^2 (G-F)^2\right]
\end{aligned}} \qquad (4.8)$$

The forms (4.7) and (4.8) are suitable for analytical examination for small and large argument values, which is done in the next section. The problem of numerical evaluation is discussed in Sec. 10.

The consistency of Method II's θ_2 - estimate in (2.5) follows immediately from the formula for the fourth moment of r given in (3.7). Indeed,

$$\alpha_4 = 2\theta_1^2(1+2\lambda+\tfrac{1}{2}\lambda^2) = 2\theta_1^2 + 4\theta_1\theta_2 + \theta_2^2 = 2\alpha_2^2 - \theta_2^2$$

because $\theta_1 + \theta_2$ is α_2.

To compute the large sample distribution we employ precisely the same technique used for Method I. Since the estimates now depend on (a_2, a_4), rather than (a_1, a_2), we have a much simpler $\underline{A}$ matrix. From (4.3) and (3.7) we have

$$\underline{A} = \begin{bmatrix} \alpha_4 - \alpha_2^2 & \alpha_6 - \alpha_2\alpha_4 \\ \alpha_6 - \alpha_2\alpha_4 & \alpha_8 - \alpha_4^2 \end{bmatrix}$$

with

$$\alpha_4 - \alpha_2^2 = \theta^2(1+2\lambda)$$

$$\alpha_6 - \alpha_2\alpha_4 = 4\theta^3(1+3\lambda+\lambda^2)$$

$$\alpha_8 - \alpha_4^2 = 4\theta^4(5+20\lambda+13\lambda^2+2\lambda^3) .$$

For the g function we take

$$g = g(x_2, x_4) = \begin{cases} (2x_2^2 - x_4)^{\frac{1}{2}} & \text{if } 2x_2^2 \geq x_4 \\ 0 & \text{otherwise} \end{cases} ,$$

so the estimates (2.5) are $\theta_1^* = a_2 - g(a_2, a_4)$ and $\theta_2^* = g(a_2, a_4)$. Again assuming $\lambda \neq 0$, the matrix of partials evaluated at the true parameter point is

$$\underline{D} = \begin{bmatrix} 1 - h_2 & h_2 \\ & \\ -h_4 & h_4 \end{bmatrix}$$

wherein

$$h_2 = \left.\frac{\partial g}{\partial x_2}\right|_{\underline{x} = \underline{\alpha}} = 2(1+\frac{1}{\lambda})$$

$$h_4 = \left.\frac{\partial g}{\partial x_4}\right|_{\underline{x} = \underline{\alpha}} = -\frac{1}{2\theta\lambda}$$

The form (4.5) again holds. The elements of $\underline{Q}$, after many frustrating polynomial manipulations, are found to be simply

$$\boxed{\begin{array}{l} \text{II.} \\ Q_{11} = \frac{1}{\lambda^2} + \frac{4}{\lambda} + 2 \\ \\ Q_{12} = -\left(\frac{1}{\lambda^2} + \frac{4}{\lambda} + 1 \right) \\ \\ Q_{22} = \frac{1}{\lambda^2} + \frac{4}{\lambda} + 1 + 2\lambda \end{array}} \qquad (4.10)$$

The determinant is

$$\boxed{\begin{array}{l} \text{II.} \\ |\underline{Q}| = \frac{1}{\lambda^2} + \frac{6}{\lambda} + 9 + 4\lambda \end{array}} \qquad (4.11)$$

The two variances and the squared correlation are graphed in Figs. 2, 3 and 4. The corresponding curves for Method I have the same qualitative properties.

5. COMPARISON OF METHODS I AND II

In this section we compare the covariance matrices (4.7) and (4.10) for small and large values of the parameter λ. The figure of comparison is the ratio of their determinants already introduced in (2.6). The determinant of a covariance matrix $\underset{\sim}{Q}$ is called the generalized variance of the distribution (centered on the origin). The reason for this terminology is that the area of the region interior to the ellipse $\underset{\sim}{x}' \underset{\sim}{Q}^{-1} \underset{\sim}{x} = c^2$, a contour of constant probability in the bivariate normal case, is $\pi c^2 |\underset{\sim}{Q}|^{\frac{1}{2}}$ (cf. Cramér [2], p. 300).

Let us examine $\underset{\sim}{Q}_I$ for small values of λ first. From (3.2)

$$I_\nu(z) \cong (z/2)^\nu/\Gamma(\nu+1) \qquad \text{as } z \to 0.$$

Referring back to (4.1), (4.2), (4.4) and (4.6) we have

$$F(0) = \sqrt{\pi}/2 \qquad G(0) = 3\sqrt{\pi}/4 \tag{5.1}$$

$$H(\lambda) \cong F'(\lambda) \cong \sqrt{\pi}\lambda/16.$$

From (4.7) we see that

$$H^2Q_{11,I} \quad , \quad -H^2Q_{12,I} \quad , \quad H^2Q_{22,I}$$

have equal limiting values as $\lambda \to 0$; viz.

$$1 - \frac{5\pi}{16} > 0.$$

We get the same thing for H^2 times (4.8), so that

$$|\underline{Q}_1| \simeq \frac{16(16-5\pi)}{\pi\lambda^2} \qquad \text{as } \lambda \to 0. \tag{5.2}$$

Combining this with (4.11) gives the value

$$d(0) = 1.49 \tag{5.3}$$

of (2.6) at the origin.

To approximate (4.7) and (4.8) when λ is large requires a bit more effort. We use the formula (3.8) with $k = 1$ and 2, and keep terms up through $1/\lambda^3$. Setting $\epsilon = 1/\lambda$ we have

$$F(\lambda) = (1+\epsilon)^{-\frac{1}{2}}\left\{1 + \frac{\Gamma^2(3/2)}{\pi^2}\left[\Gamma^2(\tfrac{1}{2})\epsilon + \tfrac{1}{2}\Gamma^2(3/2)\epsilon^2 + \tfrac{1}{6}\Gamma^2(5/2)\epsilon^3 + O(\epsilon^4)\right]\right\}$$

$$= 1 - \tfrac{1}{4}\epsilon + \frac{9}{32}\epsilon^2 - \frac{27}{128}\epsilon^3 + O(\epsilon^4) \tag{5.4}$$

and

$$G(\lambda) = (1+\epsilon)^{-\frac{3}{2}}\left\{1 + \frac{\Gamma^2(5/2)}{\pi^2}\left[\Gamma^2(-\tfrac{1}{2})\epsilon + \tfrac{1}{2}\Gamma^2(\tfrac{1}{2})\epsilon^2 + \tfrac{1}{6}\Gamma^2(3/2)\epsilon^3 + O(\epsilon^4)\right]\right\}$$

$$= 1 + \frac{3}{4}\epsilon - \frac{39}{32}\epsilon^2 + \frac{209}{128}\epsilon^3 + O(\epsilon^4)$$

as $\epsilon \to 0$. It is not true that the asymptotic expansion of $F'(\lambda)$ is given by the result of differentiating (5.4). We must return to (4.2) and use the large argument expansion [9.7.1] for I_1. In terms of ϵ we obtain for (4.6)

$$4H = \epsilon\left[1 - \frac{5}{4}\epsilon + \frac{9}{32}\epsilon^2 + O(\epsilon^3)\right].$$

Thus,

$$2(1-F^2) = \epsilon [1 - \frac{5}{4} \epsilon + \frac{9}{8} \epsilon^2 + O(\epsilon^3)]$$

$$G-F = \epsilon [1 - \frac{3}{2} \epsilon + \frac{59}{32} \epsilon^2 + O(\epsilon^3)]$$

$$2(H+\tfrac{1}{2}F) = 1 + \tfrac{1}{4}\epsilon - \frac{11}{32} \epsilon^2 + O(\epsilon^3) .$$

After some series manipulations, we find the quantities in (4.7) are in the limit the same as the corresponding ones in (4.10):

$$\lim_{\lambda \to \infty} \frac{Q_{ij,I}}{Q_{ij,II}} = 1 \qquad d(\infty) = 1 . \tag{5.5}$$

If we retain terms up through ϵ^2 in the square-bracketed quantity in (4.8), we find in terms of λ

$$|\underline{Q}_I| \cong O(1/\lambda) + 8 + 4\lambda \tag{5.6}$$

as $\lambda \to \infty$. This is to be compared with the linear approximation $9+4\lambda$ to $|\underline{Q}_{II}|$.

The determinants (4.8) and (4.11), and their ratio, are plotted in Figs. 5 and 6.

6. ASYMPTOTIC EFFICIENCY USING AMPLITUDE DATA

Let (θ_1^*, θ_2^*) be estimates of (θ_1, θ_2) based on independent observations $r_1, r_2, \ldots, r_n$ having a common probability density function f, and suppose $\sqrt{n}(\theta_1^* - \theta_1)$ and $\sqrt{n}(\theta_2^* - \theta_2)$ tend to $\underline{0}$ mean normality as $n \to \infty$ with covariance matrix $\underline{\Sigma}$. The joint asymptotic efficiency of (θ_1^*, θ_2^*) is defined as

$$e = \frac{1}{|\underline{J}| \, |\underline{\Sigma}|} \leq 1 \tag{6.1}$$

(Cramér-Rao) where

$$\underline{J} = - \left[\mathcal{E} \frac{\partial^2 \log f}{\partial \theta_i \partial \theta_j} \right] \tag{6.2}$$

is the so-called information matrix. The expectation is taken under the assumption that (θ_1, θ_2) is the true parameter point, and both $\underline{J}$ and $\underline{\Sigma}$ are presumed nonsingular.†

The regularity conditions which ensure the existence of a consistent solution $(\hat{\theta}_1, \hat{\theta}_2)$ of the Likelihood equations

$$\frac{\partial \log L}{\partial \theta_1} = 0 \qquad \frac{\partial \log L}{\partial \theta_2} = 0$$

$$L = \prod_{t=1}^{n} f(r_t ; \theta_1, \theta_2)$$

such that $\sqrt{n}(\hat{\theta}_1 - \theta_1)$ and $\sqrt{n}(\hat{\theta}_2 - \theta_2)$ are asymptotically

$$N(\underline{0}, \underline{J}^{-1})$$

are satisfied by the density (3.3). (We indifferently write $f(\cdot\, ; \theta_1, \theta_2)$ and $f(\cdot\, ; \theta, \lambda)$ for the same function.)

It is important not to infer any optimum fixed sample size properties of the Maximum Likelihood (abbrev. ML) estimate. Indeed, since

$$I_0\left(2\sqrt{\frac{\lambda}{\theta}}\, r\right) \neq H_1\left(\sqrt{\frac{\lambda}{\theta}}\right) H_2(r),$$

there exists no jointly sufficient, and hence no jointly efficient, estimate of (θ_1, θ_2) (except when $\lambda = 0$). The practical implication of this is that we are not guaranteed a unique pair of solutions of the Likelihood equations for any fixed n. However, as we shall see, there is a unique pair of nontrivial solutions.

We continue to write the various formulae in terms of

$$\theta = \theta_1 \qquad\qquad \lambda = \theta_2 / \theta_1$$

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† For the general material used in this section see Sec. 32.6 of Cramér [2] and Secs. 18.22 - 18.28 of Kendall and Stuart [4].

and introduce the abbreviation

$$R = r \frac{I_1\left(2\sqrt{\frac{\lambda}{\theta}}\, r\right)}{I_0\left(2\sqrt{\frac{\lambda}{\theta}}\, r\right)} \quad . \tag{6.3}$$

From the logarithm of (3.3) we then have

$$\theta \frac{\partial \log f}{\partial \theta_1} = -2\sqrt{\frac{\lambda}{\theta}}\, R + \frac{r^2}{\theta} + \lambda - 1$$

$$\theta \frac{\partial \log f}{\partial \theta_2} = \frac{1}{\sqrt{\theta\lambda}}\, R - 1 \qquad . \tag{6.4}$$

The Likelihood equations are

$$0 = \theta \frac{\partial \log L}{\partial \theta_1} = -2\sqrt{\frac{\lambda}{\theta}} \sum_{t=1}^{n} R_t + \frac{na_2}{\theta} + n(\lambda - 1)$$

$$0 = \theta \frac{\partial \log L}{\partial \theta_2} = \frac{1}{\sqrt{\theta\lambda}} \sum_{t=1}^{n} R_t - n$$

wherein R_t means R with r replaced by the t^{th} observed value r_t. If we multiply the second of these by 2λ and add the result to the first we obtain $\hat{\theta}(1+\hat{\lambda}) = a_2$, or

$$\hat{\theta}_1 + \hat{\theta}_2 = a_2 \tag{6.5}$$

in terms of the quantities of interest. The other Likelihood equation is

$$T(x) = \frac{1}{n} \sum_{t=1}^{n} r_t \frac{I_1\left(\frac{2r_t x}{a_2 - x^2}\right)}{I_0\left(\frac{2r_t x}{a_2 - x^2}\right)} = x. \tag{6.6}$$

Since I_1 vanishes at the origin, $x = 0$ is always a solution (which is incorrect unless $\lambda = 0$). There is one other root $x^2 = \hat{\theta}_2$ which, together with (6.5), gives a consistent asymptotically efficient pair of estimates of (θ_1, θ_2). We can approximate the desired solution by successive substitutions

$$x_{i+1} = T(x_i),$$

initialized by (say)

$$x_0 = \sqrt{\frac{\lambda^*}{1+\lambda^*}\, a_2}$$

with λ^* given in (2.3).†

We first show that $\mathcal{E}R^k$ exists for every fixed k, and in the process of so doing obtain the explicit formula for $\mathcal{E}R$. After changing the dummy of integration to $z = 2\sqrt{\frac{\lambda}{\theta}}\, r$ we have integrals depending on λ,

$$\gamma_k = \frac{(4\lambda)^{\frac{k+2}{2}} e^{\lambda}}{2\theta^{\frac{k}{2}}}\, \alpha_k = \int_0^\infty z^{k+1} e^{-\frac{z^2}{4\lambda}} I_0(z)dz$$

$$\Gamma_k = \frac{(4\lambda)^{\frac{k+2}{2}} e^{\lambda}}{2\theta^{\frac{k}{2}}}\, \mathcal{E}R^k = \int_0^\infty z^{k+1} e^{-\frac{z^2}{4\lambda}} \left[\frac{I_1(z)}{I_0(z)}\right]^{k-1} I_1(z)dz \qquad (6.7)$$

where α_k has already been calculated in Sec. 3. The ratio $I_1(z)/I_0(z)$ increases

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† This procedure might run into difficulty when the underlying parameter λ is small. A different method for approximating the nonzero root of (6.6), which circumvents this problem, is used by T. R. Benedict and T. T. Soong of Cornell Aeronautical Laboratory.

steadily from 0 to 1 with z. Thus, since the integrands are positive functions,

$$\Gamma_k \leq \int_0^\infty z^{k+1} e^{-\frac{z^2}{4\lambda}} I_1(z)dz \tag{6.8}$$

for all $k \geq 1$. Since $I_1(z)dz = dI_0(z)$, and

[9.7.1] $$I_\nu(z) \cong \frac{e^z}{\sqrt{2\pi z}} \qquad \text{as} \quad z \to \infty$$

for all orders ν, we find in the notation (6.7)

$$\int_0^\infty z^{k+1} e^{-\frac{z^2}{4\lambda}} I_1(z)dz = \frac{\gamma_{k+1}}{2\lambda} - (k+1)\gamma_{k-1} < \infty . \tag{6.9}$$

Thus, all moments of R exist. For $k=1$ there is equality in (6.8), and consequently

$$\begin{aligned} \mathcal{E} R &= \frac{2\theta^{\frac{1}{2}}}{(4\lambda)^{3/2} e^{\lambda}} \left[\frac{\gamma_2}{2\lambda} - \gamma_0 \right] \\ &= \sqrt{\theta\lambda} \quad \text{i.e.} \quad \rho \end{aligned} \tag{6.10}$$

because

$$\alpha_0 = 1 \qquad \alpha_2 = \theta(1+\lambda).$$

In the same way the change of variable and integration by parts gives

$$\begin{aligned} \mathcal{E} r^2 R &= \frac{2e^{-\lambda}}{\theta} \int_0^\infty r^4 e^{-\frac{r^2}{\theta}} I_1\left(2\sqrt{\frac{\lambda}{\theta}}\, r\right) dr \\ &= \frac{2\theta^{3/2}}{(4\lambda)^{5/2}} \left[\frac{\gamma_4}{2\lambda} - 4\gamma_2 \right] \\ &= \theta \sqrt{\theta\lambda}\,(2+\lambda) \end{aligned} \tag{6.11}$$

because

$$\alpha_4 = 2\theta^2(1+2\lambda+\tfrac{1}{2}\lambda^2).$$

From (6.4) and (6.10) we get

$$\mathcal{E}\,\frac{\partial \log f}{\partial \theta_1} = \mathcal{E}\,\frac{\partial \log f}{\partial \theta_2} = 0 \ .$$

From this it follows that (6.2) is alternatively given by

$$\underline{J} = \left[\mathcal{E}\,\frac{\partial \log f}{\partial \theta_i}\,\frac{\partial \log f}{\partial \theta_j}\right] = [J_{ij}], \tag{6.12}$$

which is easier to work with for the particular density under consideration (although in most problems (6.2) is). Using (6.10) and (6.11), together with the by now familiar formulae for α_2 and α_4, we obtain from (6.4) and (6.12)

$$\begin{aligned}
\theta^2 J_{11} &= \frac{4\lambda}{\theta}\,\mathcal{E}R^2 + 1 - 2\lambda - 4\lambda^2 &&= 1 - 2\lambda + 4\lambda^2\Phi \\
\theta^2 J_{12} &= -\frac{2}{\theta}\,\mathcal{E}R^2 + 1 + 2\lambda &&= 1 - 2\lambda\Phi \\
\theta^2 J_{22} &= \frac{1}{\theta\lambda}\,\mathcal{E}R^2 - 1 &&= \Phi \quad .
\end{aligned} \tag{6.13}$$

We have put (see (6.7))

$$\Phi + 1 = \frac{\mathcal{E}R^2}{\theta\lambda} = \frac{e^{-\lambda}}{8\lambda^3}\int_0^\infty z^3 e^{-\frac{z^2}{4\lambda}}\,\frac{I_1^2(z)}{I_0(z)}\,dz \tag{6.14}$$

which always exceeds 1 because $\theta\lambda = \mathcal{E}^2 R \leq \mathcal{E}R^2$.

Since Φ is independent of θ we can write, in keeping with our previous notation,

$$\underline{\Sigma}^{\circ} = \underline{J}^{-1} = \theta^2 \underline{Q}^{\circ}$$

under the assumption that

$$\frac{1}{|\underline{Q}^{\circ}|} = \theta^4 |\underline{J}| = (1+2\lambda)\ \Phi - 1 \neq 0. \tag{6.15}$$

We then have from (6.13)

$$\boxed{\text{ML (amplitude).} \quad \underline{Q}^{\circ} = \frac{1}{(1+2\lambda)\Phi - 1} \begin{bmatrix} \Phi & 2\lambda\,\Phi - 1 \\ 2\lambda\,\Phi - 1 & 1 - 2\lambda + 4\lambda^2\Phi \end{bmatrix}} \tag{6.16}$$

for the covariance matrix of the limiting normal distribution of $\sqrt{n}(\hat{\theta}_1 - \theta_1)/\theta_1$ and $\sqrt{n}(\hat{\theta}_2 - \theta_2)/\theta_1$. The function $\Phi = \Phi(\lambda)$ defined in (6.14) decreases steadily from 1 to 0 as λ goes from 0 to ∞. Figure 7 graphs the results of doing the integral numerically, and Fig. 8 the ensuing values of the determinant of $\underline{Q}^{\circ}$. See Appendix B and the acknowledgment contained therein.

We can check the calculations leading to $\underline{Q}^{\circ}$ by using (6.5). If we square

$$\sqrt{n}(\hat{\theta}_1 - \theta_1) + \sqrt{n}(\hat{\theta}_2 - \theta_2) = \sqrt{n}(a_2 - \alpha_2)$$

and take expectations, we have (see (4.3))

$$Q^{\circ}_{11} + 2Q^{\circ}_{12} + Q^{\circ}_{22} = 1 + 2\lambda$$

in the limit as $n \to \infty$. Upon substituting the formulae for the Q°_{ij}'s we obtain the desired redundancy, $\Phi = \Phi$.

We can deduce expressions for the elements in (6.16) when λ is large by the following trick. It was pointed out following (3.8) that

$$r \text{ is approximately } N(\rho, \tfrac{1}{2}\theta)$$

for r in the neighborhood of large ρ with θ fixed. Let us suppose we are actually sampling from this distribution. The ML estimate of $\frac{1}{2}\theta$, when the mean is unknown, is the sample variance $m_2 = a_2 - a_1^2$. As $n \to \infty$, $\sqrt{n}(m_2 - \mu_2)$ tends to be normally distributed about 0 with variance $\mu_4 - \mu_2^2$ which, by the underlying normality, is equal to $2\mu_2^2$. Since μ_2 is $\frac{1}{2}\theta$, this is tantamount to saying

$$\sqrt{n}(\hat{\theta} - \theta) \sim N(0, 2\theta^2)$$

where θ, as always, is θ_1. Referring back to (6.16), we should therefore have

$$Q^\circ_{11} = \frac{\Phi}{(1+2\lambda)\,\Phi - 1} \simeq 2.$$

Solving for Φ gives

$$\Phi \simeq \frac{2}{1+4\lambda} = \frac{1}{2\lambda} - \frac{1}{8\lambda^2} + O(\lambda^{-3}). \tag{6.17}$$

Thus

$$\underline{Q}^\circ \simeq \begin{bmatrix} 2 & -1 \\ -1 & 1+2\lambda \end{bmatrix} \qquad \text{as } \lambda \to \infty, \tag{6.18}$$

with linearly increasing determinant

$$|\underline{Q}^\circ| \simeq 1 + 4\lambda\ . \tag{6.19}$$

It is shown in Appendix B that (6.17) is indeed the case, and consequently both Methods I and II will approach full asymptotic efficiency as λ increases. If we choose $\underline{\Sigma}$ as $\underline{\Sigma}_{II} = \theta^2 \underline{Q}_{II}$ in (6.1) we obtain e, as expected, as a function only of λ:

$$e_{II}(\lambda) = \frac{|\underline{Q}^{\circ}(\lambda)|}{|\underline{Q}_{II}(\lambda)|} \quad . \tag{6.20}$$

We write the asymptotic efficiency of Method I as

$$e_{I}(\lambda) = \frac{e_{II}(\lambda)}{d(\lambda)} \ , \tag{6.21}$$

where the relative measure $d(\lambda)$ has already been shown in (5.5) to equal 1 at $\lambda = \infty$. Thus, upon comparing (6.19) and (4.11),

$$e_{I}(\infty) = e_{II}(\infty) = 1.$$

We note in passing that the estimate of θ_1 given by Method I also becomes the sample variance when we replace F by the large argument approximation

$$F(\lambda) = \sqrt{\frac{\lambda}{1+\lambda}}$$

given by (5.4). Inverting this, we have from (2.3) and (2.4)

$$\theta_1^* = \frac{a_2}{1 + \dfrac{(a_1^2/a_2)}{1-(a_1^2/a_2)}} = a_2 - a_1^2 \tag{6.22}$$

$$\theta_2^* = a_1^2 \quad .$$

For small values of the parameter we have (see Appendix B)

$$\phi = 1 - 2\lambda + 5\lambda^2 + O(\lambda^3) \tag{6.23}$$

which combines with (6.15) to give

$$\lim_{\lambda \to 0} \lambda^2 |\underline{Q}^\circ| = 1. \tag{6.24}$$

Thus, from (4.11) and (5.3)

$$e_I(0) = .68 \qquad e_{II}(0) = 1. \tag{6.25}$$

The functions (6.20) and (6.21) are graphed in Fig. 9.

7. DISTRIBUTION OF SAMPLE COVARIANCES AND CROSS COVARIANCES

We now turn to somewhat different theoretical considerations, viz. the joint large sample distribution of the random variables appearing in (2.7) and (2.12). In anticipation of future needs, we consider a pair of more general time series

$$\begin{aligned} x_t &= F_t + u_{1t} \\ y_t &= G_t + u_{2t} \end{aligned} \tag{7.1}$$

where, for notational simplicity, we allow t to range over all positive and negative integers. We take $\{u_{1t}\}$ and $\{u_{2t}\}$ to be independent and identically distributed as some zero mean linear Gaussian process, say $\{u_t\}$, with a covariance sequence

$$\sigma_h = \mathcal{E} u_t u_{t+|h|} \qquad \sum_{h=-\infty}^{\infty} |\sigma_h| < \infty \tag{7.2}$$

The following assertions, for the class of "uniformly asymptotically stationary" regression sequences, are proved in Appendix C.

Theorem. Let $\{F_t\}$ and $\{G_t\}$ in (7.1) be any two bounded real number sequences $(t = 0, \pm 1, \pm 2, \ldots)$ such that

$$\frac{1}{n}\sum_{t=1}^{n} F_t F_{t+h} = \lambda_h^{(F)} + o(1/\sqrt{n}) \tag{7.3a}$$

$$\frac{1}{n}\sum_{t=1}^{n} G_t G_{t+h} = \lambda_h^{(G)} + o(1/\sqrt{n}) \tag{7.3b}$$

$$\frac{1}{n}\sum_{t=1}^{n} F_t G_{t+h} = \mu_h + o(1/\sqrt{n}) \tag{7.3c}$$

where the order terms exist independently of h, and the λ's are necessarily even.

Define

$$A_h^{(n)} = \frac{1}{n}\sum_{t=1}^{n} x_t x_{t+h} \qquad C_h^{(n)} = \frac{1}{n}\sum_{t=1}^{n} x_t y_{t+h} . \tag{7.4}$$

Then, for every fixed integer h,

$$A_h^{(n)} \to \lambda_h^{(F)} + \sigma_h \qquad C_h^{(n)} \to \mu_h \tag{7.5}$$

as $n \to \infty$ both in mean square and with probability one. Set

$$U_h^{(n)} = \sqrt{n}(A_h^{(n)} - \lambda_h^{(F)} - \sigma_h) \qquad W_h^{(n)} = \sqrt{n}(C_h^{(n)} - \mu_h) .$$

Then

$$\lim_n \mathcal{E} U_a^{(n)} U_b^{(n)} = \sum_{k=-\infty}^{\infty} \sigma_k [2(\lambda_{k+b-a}^{(F)} + \lambda_{k+b+a}^{(F)}) + (\sigma_{k+b-a} + \sigma_{k+b+a})] \tag{7.6a}$$

$$\lim_n \mathcal{E}\, W_a^{(n)} W_b^{(n)} = \sum_{k=-\infty}^{\infty} \sigma_k [\lambda_{k+b-a}^{(F)} + \lambda_{k+b-a}^{(G)} + \sigma_{k+b-a}] \tag{7.6b}$$

$$\lim_n \mathcal{E}\, U_a^{(n)} W_b^{(n)} = \sum_{k=-\infty}^{\infty} \sigma_k [\mu_{k+b-a} + \mu_{k+b+a}] \tag{7.6c}$$

for any integers a and b, provided $\Sigma\, \sigma_k \lambda_k^{(F)}$ etc. exist. Furthermore, any collection of random variables

$$U_{h_1}^{(n)}, \ldots, U_{h_p}^{(n)}, W_{h_{p+1}}^{(n)}, \ldots, W_{h_{p+q}}^{(n)}$$

have a large sample (p+q) -variate normal distribution with zero mean vector and covariance matrix whose entries can be obtained from the formulae (7.6).

We further set

$$V_h^{(n)} = \sqrt{n}(B_h^{(n)} - \lambda_h^{(G)} - \sigma_n) \qquad B_h^{(n)} = \frac{1}{n}\sum_{t=1}^{n} y_t y_{t+h} \tag{7.7}$$

No separate proof is required to establish that

$$\lim_n \mathcal{E}\, V_a^{(n)} V_b^{(n)} = \text{(7.6a) with (F) replaced by (G)} \tag{7.8}$$

and

$$\lim_n \mathcal{E}\, U_a^{(n)} V_b^{(n)} = 0 \tag{7.9}$$

by x and y independence. Since (7.6c) does not depend on the $\lambda^{(F)}$'s which appear in the centering of the U's, it follows that

$$\lim_n \mathcal{E}\, V_a^{(n)} W_b^{(n)} = \lim_n \mathcal{E}\, U_a^{(n)} W_b^{(n)} \tag{7.10}$$

for all a, b(as can be verified by direct evaluation). V's can be added to the statement

concerning joint asymptotic normality of U's and W's, and the preceding formulae used to compute covariances of the distribution.

Specialization. Our model (1.1) is subsumed under (7.1) with

$$F_t = \rho\cos(\omega t + \varphi) \qquad G_t = \rho\sin(\omega t + \varphi)$$

$$\sigma_h = \sigma^2 \delta_{h,0}. \tag{7.11}$$

It is shown in Appendix D that the hypotheses in (7.3) hold, for any $h \gtreqless 0$, with

$$\lambda_h^{(F)} = \cos^2\varphi \cdot \rho^2 \cos h\omega \qquad \lambda_h^{(G)} = \sin^2\varphi \cdot \rho^2 \cos h\omega$$

$$\text{if } \omega = 0, \pm\pi, \pm 2\pi, \ldots$$

$$\lambda_h^{(F)} = \lambda_h^{(G)} = \tfrac{1}{2}\rho^2 \cos h\omega$$

$$\text{if } \omega \neq 0, \pm\pi, \pm 2\pi, \ldots \tag{7.12}$$

and

$$\mu_h = \begin{cases} \tfrac{1}{2}\sin 2\varphi \cdot \rho^2 \cos h\omega & \text{if } \omega = 0, \pm\pi, \pm 2\pi, \ldots \\[2ex] \tfrac{1}{2}\rho^2 \sin h\omega & \text{if } \omega \neq 0, \pm\pi, \pm 2\pi, \ldots \end{cases} \tag{7.13}$$

From our Theorem, the limits asserted in (2.7) and (2.12) are thus seen to be correct (the change of the upper limit from n to n-h in the definitions of A, B and C having no effect in the limit).

We now put

$$R_h^{(n)} = \sqrt{n}(S_h^{(n)} - \gamma_h) = U_h^{(n)} + V_h^{(n)}$$

$$T_h^{(n)} = \sqrt{n}(2C_h^{(n)} - 2\mu_h) = 2W_h^{(n)} \tag{7.14}$$

in the notation of the Theorem. Any collection of random variables

$$R_{h_1}^{(n)}, \dots, R_{h_p}^{(n)}, \ T_{h_{p+1}}^{(n)}, \dots, T_{h_{p+q}}^{(n)}$$

tends to joint normality. Using (7.6) through (7.13), we find

$$\lim_n \mathcal{E} R_a^{(n)} R_b^{(n)} = \theta^2 [\tfrac{1}{2}(\delta_{a,b} + \delta_{a,-b}) + 2\lambda \cos a\omega \cos b\omega] \qquad \text{(all } \omega)$$

$$\lim_n \mathcal{E} T_a^{(n)} T_b^{(n)} = \theta^2 [\delta_{a,b} + 2\lambda \cos(b-a)\omega] \tag{7.15}$$

and

$$\lim_n \mathcal{E} R_a^{(n)} T_b^{(n)} = \begin{cases} 2\theta^2 \lambda \cos a\omega \sin b\omega & \omega \neq 0, \pm\pi, \pm 2\pi, \dots \\ \\ 2\theta^2 \lambda \sin 2\varphi \cos a\omega \cos b\omega & \omega = 0, \pm\pi, \pm 2\pi, \dots \end{cases} \tag{7.16}$$

We see that the limiting value of $\mathcal{E} R_0^{(n)2}$ is

$$\theta^2 (1 + 2\lambda) = \alpha_2 - \alpha_1^2 ,$$

i.e. the variance of $\sqrt{n}(a_2 - \alpha_2)$. This provides a check, since S_0 is a_2 and γ_0 is $\theta_1 + \theta_2 = \alpha_2$.

8. ANALYSIS AND COMPARISON OF METHODS III AND IV

We again apply the "delta method" of Appendix A to obtain the covariance matrix $\underline{\Sigma}$ of the asymptotic normal distribution of $\sqrt{n}(\theta_1^* - \theta_1)$ and $\sqrt{n}(\theta_2^* - \theta_2)$ for the estimates (2.11). To economize on space we drop the notational distinction between variables and true parameter values made in Sec. 4. For Method III we set

$$g = g(\gamma_0, \gamma_1 \gamma_2) = \gamma_2 - 2\xi\gamma_1 + \gamma_0$$

$$\xi = \frac{\gamma_2 + \sqrt{\gamma_2^2 + 8\gamma_1^2}}{4\gamma_1}$$

where (see (2.7) - (2.9) in θ, λ notation

$$\gamma_0 = \theta(1+\lambda) \qquad \gamma_1 = \theta\lambda\xi \qquad \gamma_2 = \theta\lambda(2\xi^2 - 1) \ .$$

Then

$$\theta_1 = g = g_1 \qquad \theta_2 = \gamma_0 - g = g_2$$

and

$$\theta_1^* = g_1(S_0, S_1, S_2) \qquad \theta_2^* = g_2(S_0, S_1, S_2) \ .$$

There are here two functions of three variables. The matrix of partials is found to depend only on ξ :

$$\underline{D} = \begin{bmatrix} \dfrac{\partial g_1}{\partial \gamma_0} & \dfrac{\partial g_2}{\partial \gamma_0} \\ \dfrac{\partial g_1}{\partial \gamma_1} & \dfrac{\partial g_2}{\partial \gamma_1} \\ \dfrac{\partial g_1}{\partial \gamma_2} & \dfrac{\partial g_2}{\partial \gamma_2} \end{bmatrix} = \frac{1}{1+2\xi^2} \begin{bmatrix} 1+2\xi^2 & 0 \\ -4\xi & 4\xi \\ 1 & -1 \end{bmatrix} \ .$$

(7.15) gives the covariance matrix of the joint asymptotic normal distribution of

$$\sqrt{n}(S_0 - \gamma_0) \ , \quad \sqrt{n}(S_1 - \gamma_1) \ , \quad \sqrt{n}(S_2 - \gamma_2) \ .$$

It is

$$\underline{A} = \theta^2 \begin{bmatrix} 1+2\lambda & 2\lambda\xi & 2\lambda(2\xi^2-1) \\ 2\lambda\xi & \frac{1}{2}+2\lambda\xi^2 & 2\lambda\xi(2\xi^2-1) \\ 2\lambda(2\xi^2-1) & 2\lambda\xi(2\xi^2-1) & \frac{1}{2}+2\lambda(2\xi^2-1)^2 \end{bmatrix} .$$

After some algebra we find

$$\underline{\Sigma} = \underline{D}' \, \underline{A} \, \underline{D} = \theta^2 \underline{P} \tag{8.1}$$

where $\underline{P} = [P_{ij}]$ depends on ξ, as well as λ, but not on θ. The dependence on the former is via the value of

$$K = \frac{1}{2} \cdot \frac{1+16\xi^2}{(1+2\xi^2)^2} , \tag{8.2}$$

and the entries are simply

$$\boxed{\begin{array}{l} \text{III.} \\ P_{11} = 1+K \\ P_{12} = -K \\ P_{22} = 2\lambda + K \end{array}} \tag{8.3}$$

The generalized variance of the distribution of $\sqrt{n}(\theta_1^* - \theta_1)/\theta_1$ and $\sqrt{n}(\theta_2^* - \theta_2)/\theta_1$ is thus

$$\boxed{\begin{array}{l} \text{III.} \\ |\underline{P}| = K + 2(1+K)\lambda \end{array}} \tag{8.4}$$

The large sample distribution of the cosine estimate (2.9) is computed by the same technique. We find

$$\sqrt{n}(\xi^* - \xi) \sim N\left(0, \frac{1}{2\lambda^2} \cdot \frac{1-3\xi^2+4\xi^4}{(1+2\xi^2)^2}\right) \tag{8.5}$$

without restriction on $\xi = \cos\omega$.† λ^2 times this variance function is plotted in Fig. 10 against ξ^2. If we assume ω is not a multiple of π and invert the cosine estimate to get an estimate of frequency, the variance in (8.5) is to be divided by $\sin^2\omega$.

For the estimates (2.13) we obtain a covariance matrix which, when written in the form (8.3), has

IV.

K replaced by $L = 1 - \frac{1}{2}\xi^2 + 2\xi^2(1-\xi^2)\lambda$

$(\xi^2 \neq 1)$ (8.6)

This depends on λ as well as ξ^2.

Figure 11 is a plot of (8.2) against ξ^2. It's minimum value at 0 is $\frac{1}{2}$ and maximum at 3/8 is 8/7. Thus, from (8.4)

$$\min_{\xi} |\underline{P}_{III}| = \tfrac{1}{2} + 3\lambda \qquad \max_{\xi} |\underline{P}_{III}| = \frac{8}{7} + \frac{30}{7}\lambda . \tag{8.7}$$

It is shown in the next section that the covariance matrix of the limiting normal distribution of $\sqrt{n}(\hat{\theta}_1 - \theta_1)/\theta_1$ and $\sqrt{n}(\hat{\theta}_2 - \theta_2)/\theta_1$ is given by (8.3) with $K \equiv 0$. Thus,

$$|\underline{P}^{\circ}| = 2\lambda . \tag{8.8}$$

† The variance in (8.5) is $\frac{1}{2}$ the one given by Eq. (7.2) in [3]. This is as should be, since there we had available only a single time series, i.e. ξ^* was computed using A's rather than S's.

Figure 12 is a graph of these three straight lines.

The asymptotic efficiency of Method IV relative to Method III is less than or greater than unity depending upon the sign of the difference

$$|\underline{P}_{IV}| - |\underline{P}_{III}| = (1+2\lambda)\ (L-K). \tag{8.9}$$

The curve in Fig. 13 is

$$\lambda^{\circ} = \max\left\{0,\ \frac{K-(1-\frac{1}{2}\xi^2)}{2\xi^2(1-\xi^2)}\right\}. \tag{8.10}$$

As the figure labels indicate

$$|\underline{P}_{IV}| < |\underline{P}_{III}| \quad \text{for all } \lambda < \lambda^{\circ} \tag{8.11}$$

and conversely.

9. ASYMPTOTIC EFFICIENCY USING CARTESIAN DATA

The logarithm of the likelihood of the n pairs of observations (1.1), with a linear phase model, is

$$\log L = -n\log\pi - n\log\theta - \frac{1}{\theta} S$$

$$S = \sum_{t=1}^{n} [(x_t - \mu_{1t})^2 + (y_t - \mu_{2t})^2]$$

$$\mu_{1t} = \rho\cos(\omega t+\varphi) \qquad \mu_{2t} = \rho\sin(\omega t+\varphi). \tag{9.1}$$

In addition to (1.3), there are now two more parameters:

$$\theta_3 = \varphi \quad \theta_4 = \omega. \tag{9.2}$$

We have

$$\theta \frac{\partial \log L}{\partial \theta_1} = -n + \frac{1}{\theta} S$$

$$\theta \frac{\partial \log L}{\partial \theta_i} = 2 \sum_{t=1}^{n} \left[(x_t - \mu_{1t}) \frac{\partial \mu_{1t}}{\partial \theta_i} + (y_t - \mu_{2t}) \frac{\partial \mu_{2t}}{\partial \theta_i} \right] \qquad (i = 2, 3, 4) \qquad (9.3)$$

with

	μ_{1t}	μ_{2t}
$\frac{\partial}{\partial \theta_2}$	$\mu_{1t}/2\theta\lambda$	$\mu_{2t}/2\theta\lambda$
$\frac{\partial}{\partial \theta_3}$	$-\mu_{2t}$	μ_{1t}
$\frac{\partial}{\partial \theta_4}$	$-t\mu_{2t}$	$t\mu_{1t}$

. (9.4)

The four Likelinood equations are obtained by equating (9.3) to 0. We note that

$$\hat{\theta}_1 = \frac{1}{n} \sum_{t=1}^{n} \left[(x_t - \hat{\mu}_{1t})^2 + (y_t - \hat{\mu}_{2t})^2 \right]$$

where $\hat{\mu}$ means μ with the parameters replaced by any values for which the other three equations are satisfied. In the degenerate case of constant regression, i.e. $\omega = 0$, $\hat{\mu}_{1t} = \overline{x}$ and $\hat{\mu}_{2t} = \overline{y}$. Then

$$\hat{\theta}_1 \text{ is distributed as } \frac{\theta}{2n} \chi^2_{2(n-1)}$$

where, as always, θ is θ_1.

Each of the four random variables in (9.3) has zero expectation. The information contained in a sample $(x_1, y_1), \ldots, (x_n, y_n)$ of size n is given by the matrix

$$\underline{J}^{(n)} = \left[\mathcal{E} \frac{\partial \log L}{\partial \theta_i} \frac{\partial \log L}{\partial \theta_j} \right] = [\underline{J}_{ij}^{(n)}] . \tag{9.5}$$

In most problems we have $\underline{J}^{(n)} = n\,\underline{J}$, where $\underline{J}$ is the matrix of information contained in a single observation having the form (6. 12). Here, however, we will find that the information contained in the sample concerning the frequency ω increases like n^3 rather than n.

The random variable S in (9. 1) is distributed as $\theta \cdot \frac{1}{2} \chi^2_{2n}$, so that (see the formula preceding (3. 5))

$$\mathcal{E}\, S^k = \theta^k \frac{\Gamma(n+k)}{\Gamma(n)} .$$

Thus, after squaring the first equation in (9. 3) and taking expectations

$$\theta^2 J_{11}^{(n)} = n.$$

In view of independence, we have

$$\theta^2 J_{ii}^{(n)} = 2\theta \sum_{t=1}^{n} \left[\left(\frac{\partial \mu_{1t}}{\partial \theta_i} \right)^2 + \left(\frac{\partial \mu_{2t}}{\partial \theta_i} \right)^2 \right]$$

for i = 2, 3 and 4. Using (9. 4), and the formula for the sum of the first n squares, we obtain

$$\theta^2 J_{22}^{(n)} = \frac{n}{2\lambda}$$

$$\theta^2 J_{33}^{(n)} = 2\theta^2 \lambda n$$

$$\theta^2 J_{44}^{(n)} = \frac{1}{3}\, \theta^2 \lambda\, n(n+1)\,(2n+1).$$

It is easy to see that the covariance between the first and second line of (9.3) vanishes for each i because it involves the first and third moments of 0 mean normal variables. That is to say,

$$J_{12}^{(n)} = J_{13}^{(n)} = J_{14}^{(n)} = 0 .$$

Furthermore,

$$J_{23}^{(n)} = J_{24}^{(n)} = 0$$

since the "inner product" between the corresponding rows in (9.4) vanish. For the remaining one we find

$$\theta^2 J_{34}^{(n)} = \theta^2 \lambda n(n+1).$$

These combine to give

$$\underline{J}^{(n)} = \begin{bmatrix} \frac{n}{\theta^2} & 0 & 0 & 0 \\ 0 & \frac{n}{2\theta^2\lambda} & 0 & 0 \\ 0 & 0 & 2\lambda n & \lambda n(n+1) \\ 0 & 0 & \lambda n(n+1) & \frac{1}{3}\lambda n(n+1)(2n+1) \end{bmatrix} . \tag{9.6}$$

For the inverse we have

$$
\underline{J}^{(n)^{-1}} = \frac{1}{n}\begin{bmatrix} \theta^2 & 0 & 0 & 0 \\ 0 & 2\theta^2\lambda & 0 & 0 \\ 0 & 0 & \frac{2n+1}{\lambda(n-1)} & \frac{3}{\lambda(n-1)} \\ 0 & 0 & \frac{3}{\lambda(n-1)} & \frac{6}{\lambda(n-1)(n+1)} \end{bmatrix} . \tag{9.7}
$$

The marginal asymptotic distribution of $\sqrt{n}(\hat{\theta}_1 - \theta_1)$ and $\sqrt{n}(\hat{\theta}_2 - \theta_2)$ is normal with covariance matrix $\underline{\Sigma}^\circ$ given by the upper left hand 2×2 block of $n\underline{J}^{(n)^{-1}}$. Thus, in the notation of Sec. 8,

$$
\underline{\Sigma}^\circ = \theta^2 \, \underline{P}^\circ
$$

where

$$
\boxed{\text{ML (Cartesian)} \quad \underline{P}^\circ = \begin{bmatrix} 1 & 0 \\ 0 & 2\lambda \end{bmatrix}} \quad . \tag{9.8}
$$

This verifies (8.8).

Because of the zeroes in (9.6), the matrix (9.8) is the covariance matrix of the ML estimates of (θ_1, θ_2) when θ_3 and θ_4 are known. In fact, for an arbitrary known phase sequence $\{\varphi_t\}$ in (1.1), we have from (9.3) and (9.4)

$$
\hat{\theta}_1 = \frac{1}{n}\sum_{t=1}^{n}[(x_t - \hat{\rho}\cos\varphi_t)^2 + (y_t - \hat{\rho}\sin\varphi_t)^2]
$$

$$
\hat{\theta}_2 = \hat{\rho}^2 = [\frac{1}{n}\sum_{t=1}^{n}(x_t\cos\varphi_t + y_t\sin\varphi_t)]^2 \tag{9.9}
$$

10. NUMERICAL RESULTS AND RECOMMENDED USAGE

In this section we collect the results of desk-calculator computations of the various formulae derived in the preceding sections. On the basis of these we construct a flow chart stating which method is to be used.

Table I contains values of the asymptotic variances of and the squared correlation between

$$\sqrt{n}\left(\frac{\theta_1^* - \theta_1}{\theta_1}\right) \qquad \sqrt{n}\left(\frac{\theta_2^* - \theta_2}{\theta_1}\right)$$

for Method II defined by (4. 10). Then entries are plotted in Figs. 2, 3, and 4. As already mentioned, the corresponding quantities for Method I, as well as the method of ML, have the same appearance. The latter are relatively difficult to compute, and for those reasons we have not done the calculations.

Figures 5 through 9 are plots of the entries in Table 2. The values of the generalized variance of Method I (GVI) were obtained by means of the formula

$$\frac{16(1+\lambda)(1+2\lambda)}{\pi m_1^2} - 4\left[\left(\frac{5}{4}+5\lambda+6\lambda^2+2\lambda^3\right)\frac{m_0^2}{m_1^2} + \lambda(1+2\lambda)(3+2\lambda)\frac{m_0}{m_1} + 2\lambda^2(1+\lambda)\right] \tag{10. 1}$$

starting with 8 significant digits for

$$m_\nu = e^{-\frac{\lambda}{2}} I_\nu\left(\frac{\lambda}{2}\right) \qquad (\nu = 0, 1)$$

given in Table 9. 8 of the handbook [1] . The arguments $\lambda/2$ for the * values are not in this table. These were computed from the formula (4. 8) as written, starting with 6 digits for F, F′ and G which resulted from single precision 7094 computations. By comparison with the minimum 4 digit accuracy (usually 5) of (10. 1), it appears this procedure is satisfactory for plotting purposes up to $\lambda = 10$. However, for $\lambda = 16$, only 2 good digits remain and for $\lambda = 20$ but 1.

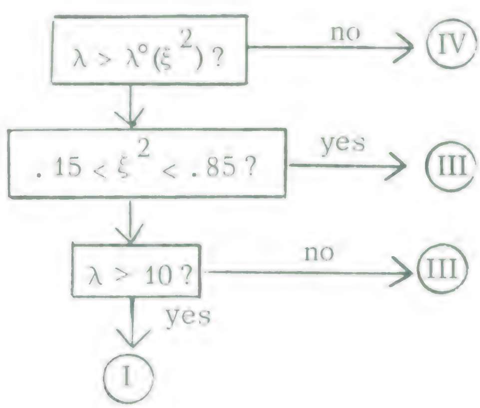
The function GVI in Fig. 5 has the proper large argument slope, 4λ, but a graphical intercept of 3. Evidently, then $\lambda = 20$ is not sufficiently large for the asymptotic linear approximation (5.6) to be of use.

The values of GVML were obtained from Benedict's numerical integration of (B2) to give $\Phi + 1$ as a function of $\alpha = \sqrt{2\lambda}$.

The entries in Table 3 are respectively plotted in Figs. 10, 11 and 13.

Table 4 gives some values of (8.4). The smaller numbers in parentheses are those with K redefined by (8.6). This reduction in generalized variance obtains if we switch from Method III to Method IV and the point (ξ^2, λ) falls below the boundary curve in Fig. 13.

The generalized variances of Methods I and II depend on $\lambda = \theta_2/\theta_1$, and those of Methods III and IV on $\xi = \cos\omega$ as well as λ. If we knew these values (which we don't) we could specify which procedure should be used. It is nonetheless reasonable to make a choice on the basis of sample quantities. In the flow chart for coherent data presented below, one thus uses as values of ξ^2 and λ estimates computed from the data via Method III, i.e. from (2.9) and the ratio of the two estimates in (2.11)).† The function λ° of ξ^2 is defined by (8.10) and (8.2). The function K



(10.2)

exceeds 1 on the interval (.15, .85). If such is the case, we have assumed the least favorable situation, viz. that K is maximum, and switched to Method I if λ exceeds the

† We could of course set up a statistical decision procedure using the large sample distributions of the estimates of ξ^2 and λ.

value at which the line $GVIII_{max}$ passes up through GVI (approximately 10 as shown in Fig. 5). From Table 4 we see that the inclusion of Method IV gives a relatively small improvement over Method III.

For incoherent data we use

$$\begin{array}{lll} \text{I} & \text{if} & \lambda > .4 \\ \text{II} & \text{if} & \lambda < .4 \end{array}$$

with λ computed as the ratio θ_2^*/θ_1^* via (2.4) or (2.5). (10.3)

Table 1. The Variances and Squared Correlation for Method II.

Note: The correlation is the negative square root of the entries in the third column.

λ	Q_{11}	Q_{22}	$\frac{Q_{12}^2}{Q_{11}Q_{22}}$
.1	142.00	141.20	.99
.2	47.00	46.40	.97
.3	26.44	26.04	.94
.4	18.25	18.05	.90
.5	14.00	14.00	.86
.8	8.56	9.16	.73
1	7.00	8.00	.64
1.5	5.11	7.11	.47
2	4.25	7.25	.34
2.5	3.76	7.76	.26
3	3.44	8.44	.21
4	3.06	10.06	.14
5	2.84	11.84	.10
6	2.69	13.69	.078
7	2.59	15.59	.063
8	2.52	17.52	.052
9	2.46	19.46	.044
10	2.41	21.41	.039
12	2.34	25.34	.033
14	2.29	29.29	.025
16	2.25	33.25	.021
18	2.23	37.23	.018
20	2.20	41.20	.016
24	2.17	49.17	.013
28	2.14	57.14	.011
32	2.13	65.13	.0092

Table 2. Generalized Variances of Methods I and II and Their Efficiencies

λ	I	II	I/II	Φ	ML	ML/I	ML/II
.1	214.5*	169.4	1.27	.83836	165.8	.773	.979
.2	73.5	64.8	1.13	.72594	61.3	.834	.946
.4	31.3	31.9	.981	.57522	28.3	.904	.887
.5	25.2*	27.0	.933	.52145	23.3	.925	.863
1	16.2	20.0	.810	.35442	15.8	.975	.790
2	15.3	20.3	.753	.21316	15.2	.994	.749
3	17.5	23.1	.759	.15101	17.5	1.000	.758
4	20.7	26.6	.779	.11648	20.7		.778
4.5	22.4*			.10446	22.4		
5	24.2	30.2	.801	.094664	24.2		.801
6	27.9	34.0	.821	.079677	27.9		.821
8	35.6	41.8	.853	.060475	35.6		.852
10	43.5	49.6	.876	.048714	43.5		.877
12.5				.039182	53.4		
16	67.3	73.4	.912	.030753	67.3		.917
20	83.2	89.3	.932	.024682	83.6		.936
24	99.2	105.3	.942	.020613	99.6		.946
30	123.1	129.2	.952				
40	162.9	169.1	.963				

Table 3. Functions Associated with Methods III and IV.

ξ^2	$\lambda^2 \cdot \mathrm{Var}\sqrt{n}(\xi^* - \xi)$	K	λ°
0	.500	.500	0
.05		.744	0
.1	.257	.903	0 ←†
.15		1.006	.318
.2	.143	1.071	.534
.25		1.111	.629
.3	.0898	1.133	.674
.375		1.143(max)	.704
.4	.0679	1.142	.713
.5	.0625(min)	1.125	.750
.6	.0661	1.095	.823
.7	.0747	1.059	.974
.8	.0858	1.021	1.32
.85		1.001	1.671
.9	.0982	.982	2.40
.95		.963	4.611
1	.111	.944	∞

† From Fig. 11, $K = 1 - \frac{1}{2}\xi^2$ at $\xi^2 = .115$.

Table 4. Generalized Variance of Method III and Method IV in Parentheses.

Note: 2λ is the Generalized Variance of the Method of Maximum Likelihood.

2λ \ ξ^2	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1	2.00	2.80	3.14 (3.12)	3.26 (3.22)	3.28 (3.08)	3.26 (3.00)	3.20 (2.88)	3.12 (2.72)	3.04 (2.52)	2.96 (2.28)
2	3.50	4.70	5.2	5.39	5.42	5.39	5.30	5.18	5.06 (4.76)	4.94 (4.19)
3	5.00	6.60	7.3	7.52	7.56	7.52	7.40	7.24	7.08	6.92 (6.28)
4	6.50	8.50	9.4	9.65	9.70	9.65	9.50	9.30	9.10	8.90 (8.55)
5	8.00	10.40	11.42	11.78	11.84	11.78	11.60	11.36	11.12	10.88
6	9.50	12.30	13.49	13.91	13.98	13.91	13.70	13.42	13.14	12.86
7	11.00	14.20	15.56	16.04	17.12	16.04	15.80	15.48	15.16	14.84
8	12.50	16.10	17.63	18.17	18.26	18.17	17.90	17.54	17.18	16.82
9	14.00	18.00	19.70	20.30	20.40	20.30	20.00	19.60	19.20	18.80
10	15.50	19.90	21.77	22.43	22.54	22.43	22.10	21.66	21.22	20.78
20	31.5	39.9	43.5	44.7	49.9	44.7	44.3	42.4	41.6	40.7

APPENDIX A

THE DELTA METHOD

A standard technique in large sample distribution theory is called the "delta method." We state it here in general form, since it is used throughout the paper.

Suppose $\underline{t}_1, \underline{t}_2, \ldots$ is a sequence of estimates of a p-dimensional parameter $\underline{\theta}$ such that the distribution of $\sqrt{n}(\underline{t}_n - \underline{\theta})$ converges as $n \to \infty$ to that of a p-variate normal random variable with mean vector $\underline{0}$ and covariance matrix $\underline{A}$ (not necessarily of full rank). We express this in symbols by

$$\sqrt{n}(\underline{t}_n - \underline{\theta}) \sim N(\underline{0}, \underline{A}).$$

Let g_j be a function of p variables such that

$$d_{ij}(\underline{x}) = \frac{\partial g_j(x_1, x_2, \ldots, x_p)}{\partial x_i} \qquad (i = 1, 2, \ldots, p)$$

are continuous at $\underline{x} = \underline{\theta}$ with at least one nonzero for each $j = 1, 2, \ldots, q$. Put

$$\underline{D} = [\, d_{ij}(\underline{\theta}) : i = 1, 2, \ldots, p \, ; \; j = 1, 2, \ldots, q]$$

and let $\underline{g}(\underline{x})$ be the column q-vector with j component $g_j(\underline{x})$. Then

$$\sqrt{n}(\underline{g}(\underline{t}_n) - \underline{g}(\underline{\theta})) \sim N(\underline{0}, \underline{D}' \underline{A} \underline{D}).$$

The conditions of this theorem place no requirement on the existence of moments. In our applications, however, the ij^{th} entry of $\underline{A}$ is the limiting value of $n\mathcal{E}(\underline{t}_n - \underline{\theta})(\underline{t}_n - \underline{\theta})'$.

APPENDIX B

AN INTEGRAL

In Rice's notation,

$$\alpha = \rho/\sigma \qquad \text{i.e.} \quad \alpha^2 = 2\lambda, \tag{B. 1}$$

the integral (6. 14) is

$$\Phi + 1 = \int_0^\infty \left(\frac{x}{\alpha}\right)^2 \frac{I_1^2(dx)}{I_0^2(\alpha x)} \cdot x\, e^{-\frac{1}{2}(x^2+\alpha^2)} I_0(\alpha x)\,dx. \tag{B. 2}$$

The function following the center dot is the density of $x = r/\sigma$, i.e. a χ_2' variate. In terms of λ we have the small and large argument expansions

$$\Phi = \begin{cases} 1 - 2\lambda + 5\lambda^2 + O(\lambda^3) & \text{as } \lambda \to 0 \\ \dfrac{1}{2\lambda} - \dfrac{1}{8\lambda^2} + O\left(\dfrac{1}{\lambda^3}\right) & \text{as } \lambda \to \infty . \end{cases} \tag{B. 3}$$

The approximation

$$\Phi \approx \frac{1}{1+2\lambda} \tag{B. 4}$$

agrees with the result of numerical integration to 2 digits over the entire range[†]. Referring to (6. 15), we see that the information matrix will therefore be close to singularity.

† The author wishes to thank T. R. Benedict of the Cornell Aeronautical Laboratory for the results of his investigation of the integral $\Phi + 1$ and the values obtained by numerical integration.

APPENDIX C

PROOF OF THE THEOREM OF SECTION 7

We set

$$X_h^{(n)} = \frac{1}{n}\sum_{t=1}^{n} F_t u_{t+h} \qquad Y_h^{(n)} = \frac{1}{n}\sum_{t=1}^{n} F_{t+h} u_t$$

$$Z_h^{(n)} = \frac{1}{n}\sum_{t=1}^{n} (u_t u_{t+h} - \sigma_h)$$

and drop the subscript 1 from u_{1t} since for the present we are dealing only with $\{x_t\}$. By the assumption (7.6a)

$$A_h^{(n)} - \lambda_h^{(F)} - \sigma_h = X_h^{(n)} + Y_h^{(n)} + Z_h^{(n)} + o(1/\sqrt{n}). \tag{C1}$$

The arithmetic means are centered at expectations and the order term is a sure one. (7.5) is an almost immediate consequence of the following Law of Large Numbers for dependent random variables (see Parzen [6], p. 419) : Let

$$\bar{v}_n = \frac{1}{n}\sum_{t=1}^{n} v_t$$

where $v_1, v_2, \ldots$ is any sequence of 0 mean random variables having uniformly bounded variances. Then as $n \to \infty$

$$\mathcal{E}\bar{v}_n^2 \to 0 \qquad \text{iff} \qquad \mathcal{E}\bar{v}_n v_n \to 0$$

$$\bar{v}_n \to 0 \text{ a.s.} \qquad \text{if} \qquad \mathcal{E}\bar{v}_n v_n = O(1/n^{\delta}). \quad \text{for some } \delta > 0.$$

We have

$$|\mathcal{E} X_h^{(n)} F_n u_{n+h}| = |\frac{1}{n}\sum_{t=1}^{n} F_t F_n \sigma_{n-h}| \le \frac{\text{const.}}{n}\sum_{t=0}^{n-1}|\sigma_t| = O(1/n)$$

by (7.2), and similarly for Y. For Gaussian processes

$$\mathcal{E} u_t u_{t+h_1} u_{t+h_2} u_{t+h_3} = \sigma_{h_1}\sigma_{h_2-h_3} + \sigma_{h_2}\sigma_{h_1-h_3} + \sigma_{h_3}\sigma_{h_1-h_2} . \tag{C2}$$

Thus

$$\mathcal{E} Z_h^{(n)}(u_n u_{n+h} - \sigma_h) = \frac{1}{n}\sum_{t=1}^{n}(\mathcal{E} u_t u_{t+h} u_n u_{n+h} - \sigma_h^2)$$

$$= \frac{1}{n}\sum_{t=1}^{n}(\sigma_{n-t}^2 + \sigma_{n-t+h}\sigma_{n-t-h}) ,$$

and

$$|\mathcal{E} Z_h^{(n)}(u_n u_{n+h} - \sigma_h)| \le \frac{1}{n}\sum_{t=0}^{n-1}(\sigma_t^2 + |\sigma_{t+h}\sigma_{t-h}|) = O(1/n)$$

by the Schwarz Inequality and square summability of $\{\sigma_h\}$. This proves the first statement in (7.5).

Multiplying (C1) through by $\sqrt{n}$ gives

$$U_h^{(n)} = \sqrt{n}\, X_h^{(n)} + \sqrt{n}\, Y_h^{(n)} + \sqrt{n}\, Z_h^{(n)} + o(1) . \tag{C3}$$

Since for 0 mean Gaussian processes

$$\mathcal{E} u_t u_{t+h_1} u_{t+h_2} = 0 \tag{C4}$$

for all indices, we have

$$n\,\mathcal{E}X_a^{(n)}Z_b^{(n)} = \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n} F_t(\mathcal{E}u_{t+a}u_s u_{s+b} - \mathcal{E}u_{t+a}\sigma_b) = 0 .$$

Similarly, $n\,\mathcal{E}Y_a^{(n)}Z_b^{(n)} = 0$ for all a, b. Thus,

$$\mathcal{E}U_a^{(n)}U_b^{(n)} = n\,\mathcal{E}[X_a^{(n)}X_b^{(n)} + Y_a^{(n)}Y_b^{(n)}]$$

$$+ n\,\mathcal{E}[X_a^{(n)}Y_b^{(n)} + X_b^{(n)}Y_a^{(n)}]$$

$$+ n\,\mathcal{E}Z_a^{(n)}Z_b^{(n)} + o(1) . \tag{C5}$$

The limiting value of the penultimate is easily evaluated. We have from (C2)

$$n\,\mathcal{E}Z_a^{(n)}Z_b^{(n)} = \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}(\mathcal{E}u_t u_{t+a}u_s u_{s+b} - \sigma_a\sigma_b)$$

$$= \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}(\sigma_{s-t}\sigma_{s-t+b-a} + \sigma_{s-t+b}\sigma_{s-t-a}) .$$

There are $n-|k|$ indices $t, s = 1, 2, \ldots, n$ for which the difference $s-t$ equals k ($k = 0, \pm 1, \pm 2, \ldots, \pm(n-1)$). Hence as $n \to \infty$

$$n\,\mathcal{E}Z_a^{(n)}Z_b^{(n)} = \sum_{|k| \leq n-1}\left(1-\frac{|k|}{n}\right)(\sigma_k\sigma_{k+b-a} + \sigma_{k+b}\sigma_{k-a})$$

$$\to \sum_{k=-\infty}^{\infty}(\sigma_k\sigma_{k+b-a} + \sigma_{k+b}\sigma_{k-a})$$

$$= \sum_{k=-\infty}^{\infty}\sigma_k(\sigma_{k+b-a} + \sigma_{k+b+a}) \tag{C6}$$

by the Kronecker Lemma and the summability of $\{\sigma_k^2\}$. (We repeatedly use the fact that $\sigma_{-k} = \sigma_k$ to rewrite double infinite summations.)

The main burden of the entire derivation of (7.6) is contained in the evaluation of the limit

$$L(a, b) = \lim_n n\mathcal{E}Y_a^{(n)}Y_b^{(n)} = \lim_n \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n} F_{t+a}F_{s+b}\sigma_{s-t} \tag{C7}$$

for arbitrary a and b. By appropriate changes of the summation indices we see that

$$\lim_n n\mathcal{E}X_a^{(n)}X_b^{(n)} = L(-a, -b)$$

$$\lim_n n\mathcal{E}X_a^{(n)}Y_b^{-(n)} = L(-a, b) \qquad \lim_n n\mathcal{E}X_b^{(n)}Y_a^{(n)} = L(a, -b) \tag{C8}$$

since L is symmetric.

We compute L by the following method. Let $m = m(n)$ tend to infinity over positive integers in such a way that

$$m^2/n \to 0.$$

Introduce

$$\sigma_k^{(n)} = \begin{cases} \sigma_k & \text{if} \quad |k| \le m \\ 0 & \text{if} \quad |k| > m \end{cases}$$

and

$$L_n(a, b) = \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n} F_{t+a}F_{s+b}\sigma_{s-t}^{(n)}.$$

We have

$$|n\mathcal{E}Y_a^{(n)}Y_b^{(n)} - L_n(a,b)| \le \frac{\text{const.}}{n}\sum_{t=1}^{n}\sum_{s=1}^{n} |\sigma_{s-t} - \sigma_{s-t}^{(n)}|$$

$$= \text{const.} \sum_{|k|\le n-1} \left(1-\frac{|k|}{n}\right)|\sigma_k - \sigma_k^{(n)}|$$

$$= \text{const.} \sum_{m+1 \le |k| \le n-1} \left(1-\frac{|k|}{n}\right)|\sigma_k| = o(1)$$

as $n > m \to \infty$ by (7. 2) and Cauchy's criterion. Thus

$$L(a,b) = \lim_n L_n(a,b).$$

With a and b understood, we substitute $k+t$ for s in L_n and get

$$L_n = \frac{1}{n}\sum_{t=1}^{n} F_{t+a}\ell_t^{(n)} \qquad \ell_t^{(n)} = \sum_{k=-t+1}^{n-t} F_{k+t+b}\sigma_k^{(n)} .$$

n will certainly exceed $2m$ for all large values of the former, and we then find by the definition of $\{\sigma_k^{(n)}\}$

$$\ell_t^{(n)} = \begin{cases} \sum\limits_{-t+1}^{m} & \text{if } 1 \le t \le m \\ \sum\limits_{k=-m}^{m} F_{k+t+b}\sigma_k & \text{if } m+1 \le t \le n-m \\ \sum\limits_{-m}^{n-t} & \text{if } n-m+1 \le t \le n \end{cases}$$

Thus

$$L_n = \frac{1}{n} \sum_{t=m+1}^{n-m} F_{t+a} \sum_{|k| \leq m} F_{k+t+b} \sigma_k + \frac{1}{n} \left(\sum_{t=1}^{m} + \sum_{t=n-m+1}^{n} \right) F_{t+a} \ell_t^{(n)} .$$

In the first and third lines in the preceding display there are at most $2m$ summands each under uniform bound, i.e. both $\max_{1 \leq t \leq m} |\ell_t^{(n)}|$ and $\max_{n-m+1 \leq t \leq n} |\ell_t^{(n)}|$ are $\leq$ const. m.

Thus, after interchanging the order of summation,

$$L_n = \sum_{|k| \leq m} \sigma_k \left[\frac{1}{n} \sum_{t=m+1}^{n-m} F_{t+a} F_{t+k+b} \right] + O\left(\frac{m^2}{n} \right) .$$

We split up the sum within square brackets into

$$\sum_{t=1}^{n} - \sum_{t=1}^{m} - \sum_{t=n-m+1}^{n} .$$

Both the second and third of these is in absolute value $\leq$ const. m where "const." does not depend on k. Hence

$$L_n = \sum_{|k| \leq m} \sigma_k \left[\frac{1}{n} \sum_{t=1}^{n} F_{t+a} F_{t+k+b} \right] + O\left(\frac{m}{n} \sum_{|k| \leq m} |\sigma_k| \right) + O\left(\frac{m^2}{n} \right)$$

$$= \sum_{|k| \leq m} \sigma_k \left[\frac{1}{n} \sum_{t=1}^{n} F_t F_{t+k+b-a} \right] + o(1)$$

by (7.2) and the choice of m. It is at this point that uniformity in h of the order term hypothesized in (7.3a) is needed. Accordingly, the average remaining within square brackets differs in absolute value from $\lambda_{k+b-a}^{(F)}$ by a term $o(1/\sqrt{n})$ which does not depend

on k. Again, using the absolute summability of $\{\sigma_k\}$ and the fact that $m/n \to 0$, we therefore have

$$L_n = \sum_{|k| \leq m} \sigma_k \lambda^{(F)}_{k+b-a} + o(1)$$

as $n \to \infty$. Thus,

$$L(a, b) = \sum_{k=-\infty}^{\infty} \sigma_k \lambda^{(F)}_{k+b-a} \tag{C9}$$

for any fixed integers a and b.

With this result, together with (C6) and (C8), we return to (C5) and take limits as $n \to \infty$. Since λ_{-k} and λ_k and $\sigma_{-k} = \sigma_k$, we have from (C9)

$$L(-a, -b) = L(a, b) \qquad\qquad L(a, -b) = L(-a, b).$$

Consequently,

$$\lim_n \mathcal{E} U_a^{(n)} U_b^{(n)} = 2 \sum_{k=-\infty}^{\infty} \sigma_k \lambda^{(F)}_{k+b-a} + 2 \sum_{k=-\infty}^{\infty} \sigma_k \lambda^{(F)}_{k+b+a} + \text{(C6)},$$

which is the assertion (7.6a).

The preceding suffices to prove the balance of the theorem with a minimum of effort. We redefine X, Y and Z by

$$X_h^{(n)} = \frac{1}{n} \sum_{t=1}^{n} F_t u_{2,t+h} \qquad\qquad Y_h^{(n)} = \frac{1}{n} \sum_{t=1}^{n} G_{t+h} u_{1t}$$

$$Z_h^{(n)} = \frac{1}{n} \sum_{t=1}^{n} u_{1t} u_{2,t+h},$$

each of which has mean 0 because $\{u_{1t}\}$ and $\{u_{2t}\}$ have 0 mean and are independent. From (7.3c) we have

$$C_h^{(n)} - \mu_h = X_h^{(n)} + Y_h^{(n)} + Z_h^{(n)} + o(1/\sqrt{n}) .$$

The conditions of the Law of Large Numbers are again satisfied, proving the second statement in (7.5). As far as cross-moments go, not only is $\mathcal{E} X_a^{(n)} Z_b^{(n)} = \mathcal{E} Y_a^{(n)} Z_b^{(n)} = 0$, but now also

$$\mathcal{E} X_a^{(n)} Y_b^{(n)} = 0$$

for all a and b. Thus,

$$\mathcal{E} W_a^{(n)} W_b^{(n)} = n \mathcal{E} X_a^{(n)} X_b^{(n)} + n \mathcal{E} Y_a^{(n)} Y_b^{(n)} + n \mathcal{E} Z_a^{(n)} Z_b^{(n)} + o(1).$$

We have

$$\begin{aligned}
n \mathcal{E} Z_a^{(n)} Z_b^{(n)} &= \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} \mathcal{E} u_{1t} u_{2,t+a} u_{1s} u_{2,s+b} \\
&= \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} \sigma_{s-t} \sigma_{s-t+b-a} \\
&= \sum_{|k| \leq n-1} (1 - \frac{|k|}{n}) \sigma_k \sigma_{k+b-a} \\
&\to \sum_{k=-\infty}^{\infty} \sigma_k \sigma_{k+b-a} .
\end{aligned}$$

Using (7.3a) and (7.3b), and the assumption of a common covariance sequence, our previous evaluation of the limit L and (C8) combine to give

$$n \lim_{n} \mathcal{E} X_a^{(n)} X_b^{(n)} = \sum_{k=-\infty}^{\infty} \sigma_k \lambda_{k+b-a}^{(F)}$$

$$n \lim_{n} \mathcal{E} Y_a^{(n)} Y_b^{(n)} = \sum_{k=-\infty}^{\infty} \sigma_k \lambda_{k+b-a}^{(G)} \quad .$$

These prove (7.6b).

We next have, after returning the subscript 1 in (C3),

$$U_a^{(n)} W_b^{(n)} = \frac{1}{n} [\Sigma F_t u_{1,t+a} + \Sigma F_{t+a} u_{1t} + \Sigma (u_{1t} u_{1,t+a} - \sigma_a)]$$

$$\times [\Sigma F_t u_{2,t+b} + \Sigma G_{t+b} u_{1t} + \Sigma u_{1t} u_{2,t+b}] + o(1)$$

where all summations are on t from 1 to n. After taking expectations we are left with only two terms, viz. the product of the first in the first line with the second in the second plus the second with the second. Thus,

$$\mathcal{E} U_a^{(n)} W_b^{(n)} = \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} F_t G_{s+b} \sigma_{s-t-a} + \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} F_{t+a} G_{s+b} \sigma_{s-t} + o(1)$$

$$= \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} F_{t-a} G_{s+b} \sigma_{s-t} + \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} F_{t+a} G_{s+b} \sigma_{s-t} + o(1) \; .$$

We return to (C7), and its sequel, and everywhere replace F_{s+b} by G_{s+b}. The argument results in

$$\frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n} F_{t+a}G_{s+b}\sigma_{s-t} = \sum_{|k|\le m}\sigma_k\left[\frac{1}{n}\sum_{t=1}^{n}F_t G_{t+k+b-a}\right] + o(1)$$

$$\to \sum_{k=-\infty}^{\infty}\sigma_k\mu_{k+b-a}$$

by (7.3c). The previous two statements prove (7.6c).

The asymptotic normality follows immediately. Indeed, both the $\sqrt{n}\, X_h^{(n)}$'s and $\sqrt{n}\, Y_h^{(n)}$'s are normal for every n, and the $\sqrt{n}\, Z_h^{(n)}$'s (again for either definition) are known to be normal about 0 in large samples with the covariances of the limiting distribution equal to the limit of the corresponding covariances.

Extension. If we drop the assumption that $\{u_t\}$ is Gaussian, but retain that of summable second and fourth order moment sequences, as well as zero first and third, very little is changed. The only place where (C2) explicitly entered was in the evaluation of $\lim \mathcal{E} U_a^{(n)} U_b^{(n)}$. In the non-Gaussian case, the difference between the two sides of (C2) is the fourth order cumulant sequence of $\{u_t\}$,

$$\mu_{NG}(h_1, h_2, h_3) = \mathcal{E} u_t u_{t+h_1} u_{t+h_2} u_{t+h_3}$$
$$- (\sigma_{h_1}\sigma_{h_2-h_3} + \sigma_{h_2}\sigma_{h_1-h_3} + \sigma_{h_3}\sigma_{h_1-h_2}).$$

The effect of this is to add

$$\sum_{k=-\infty}^{\infty}\mu_{NG}(a, k, k+b)$$

to the right side of (C6). Suppose we write $\{u_t\}$ as a linear process, with summable impulse response coefficients, in independent identically distributed random variables,

say $\{v_t\}$, where

$$\mathcal{E} v_t = \mathcal{E} v_t^3 = 0$$

$$\mathcal{E} v_t^2 = \kappa_2 \qquad \mathcal{E} v_t^4 = \kappa_4 + 3\kappa_2^2 \quad .$$

We can then evaluate the sum to get

$$\sum_{k=-\infty}^{\infty} \mu_{NG}(a, k, k+b) = \frac{\kappa_4}{\kappa_2^2} \sigma_a \sigma_b .$$

The ratio of cumulants κ_4/κ_2^2 is the so-called coefficient of excess of $\{v_t\}$ over a $N(0, \kappa_2)$ distribution, and it vanishes when $\{v_t\}$ is so distributed.

In summary, then, if we drop the normality assumption, but retain (C4), then $\frac{\kappa_4}{\kappa_2^2}\sigma_a \sigma_b$ is to be added to (7.6a), and (7.6b) and (7.6c) are to be left unaltered.

APPENDIX D

THE UNIFORM ASYMPTOTIC STATIONARITY OF SINUSOIDAL REGRESSIONS

We may obviously ignore the multiplier ρ in (7.11). Then, using standard trigonometric additional formulae, for arbitrary integers t and h

$$
\begin{aligned}
F_t F_{t+h} &= [\cos\omega t\cos\varphi - \sin\omega t\sin\varphi]\,[\cos\omega(t+h)\cos\varphi - \sin\omega t(t+h)\sin\varphi] \\
&= \cos^2\varphi\cos\omega(t+h)\cos\omega t + \sin^2\varphi\sin\omega(t+h)\sin\omega t \\
&\quad - \sin\varphi\cos\varphi[\sin\omega(t+h)\cos\omega t + \cos\omega(t+h)\sin\omega t] \\
&= \tfrac{1}{2}\cos^2\varphi[\cos\omega h + \cos(2\omega t + \omega h)] + \tfrac{1}{2}\sin^2\varphi[\cos\omega h - \cos(2\omega t + \omega h)] \\
&\quad - \sin\varphi\cos\varphi\sin(2\omega t + \omega h) \quad . \qquad \text{(D1)}
\end{aligned}
$$

If ω is any multiple of π, the second and third terms in the last-written expression vanish by 2π-periodicity and $\sin n\pi = 0$. The first gives

$$
\frac{1}{n}\sum_{t=1}^{n} F_t F_{t+h} = \cos^2\varphi\cos\omega h \qquad (\omega = 0, \pm\pi, \pm 2\pi, \ldots) \qquad \text{(D2)}
$$

for every n. In the contrary case, we write (D1) as

$$
\begin{aligned}
F_t F_{t+h} &= \tfrac{1}{2}\cos\omega h + \tfrac{1}{2}(\cos^2\varphi - \sin^2\varphi)\cos(2\omega t + \omega h) - \sin\varphi\cos\varphi\sin(2\omega t + \omega h) \\
&= \tfrac{1}{2}\cos\omega h + \tfrac{1}{2}\cos 2\varphi\cos(2\omega t + \omega h) - \tfrac{1}{2}\sin 2\varphi\sin(2\omega t + \omega h) \\
&= \tfrac{1}{2}\cos\omega h + \tfrac{1}{2}\cos(2\omega t + \psi_h) \qquad \text{(D3)}
\end{aligned}
$$

where

$$
\psi_h = \omega h + 2\varphi
$$

doesn't contain t. Now we have the closed form summations (Knopp [5] , p. 480)

$$\sum_{t=1}^{n} \cos 2\omega t = \frac{\sin(2n+1)\omega}{2\sin\omega} - \tfrac{1}{2} = \frac{\sin n\omega \cos(n+1)\omega}{\sin\omega}$$

$$\sum_{t=1}^{n} \sin 2\omega t = \frac{\cos\omega - \cos(2n+1)\omega}{2\sin\omega} = \frac{\sin n\omega \sin(n+1)\omega}{\sin\omega} \quad . \tag{D4}$$

Both of these remain bounded as $n \to \infty$ when ω is not a multiple of π. Thus, after averaging (D3),

$$\frac{1}{n}\sum_{t=1}^{n} F_t F_{t+h} = \tfrac{1}{2}\cos\omega h + \frac{K_n}{n} \tag{D5}$$

where $K_n = O(1)$ depends on h only through $\cos\psi_h$ and $\sin\psi_h$, and hence can be bounded independent of the lag. This together with (D2) verifies (7. 3a) with $\lambda_h^{(F)}$ given by (7. 12).

Since

$$G_t = \rho\cos(\omega t + \varphi - \frac{\pi}{2})$$

we need only replace φ by $\varphi - \frac{\pi}{2}$ in the above argument to obtain the formula for $\lambda_h^{(G)}$.

For μ_h we proceed in the same way. Instead of (D1) we have

$$F_t G_{t+h} = \tfrac{1}{2}\cos^2\varphi[\sin\omega h + \sin(2\omega t + \omega h)] + \tfrac{1}{2}\sin^2\varphi[\sin\omega h - \sin(2\omega t + \omega h)]$$

$$+ \sin\varphi\cos\varphi\cos(2\omega t + \omega h),$$

and hence

$$\frac{1}{n}\sum_{t=1}^{n} F_t G_{t+h} = \sin\varphi\cos\varphi\cos\omega h = \tfrac{1}{2}\sin 2\varphi\cos\omega h \qquad (\omega = 0, \pm\pi, \pm 2\pi, \dots) \tag{D6}$$

for all n. The expression corresponding to (D3) is

$$F_t G_{t+h} = \tfrac{1}{2} \sin \omega h + \tfrac{1}{2} \sin(2\omega t + \psi_h)$$

with ψ_h unchanged. Thus, for the same reasons,

$$\frac{1}{n} \sum_{t=1}^{n} F_t G_{t+h} = \tfrac{1}{2} \sin \omega h + O(1/n) \qquad (\omega \neq 0, \pm\pi, \pm 2\pi, \ldots) \tag{D7}$$

where the order term can be taken independent of h.

FIGURE CAPTIONS†

† In Figs. 2, 3, 5, 8 and 12 the word "variance(s)" should be qualified by "after division by $4\sigma^4$."

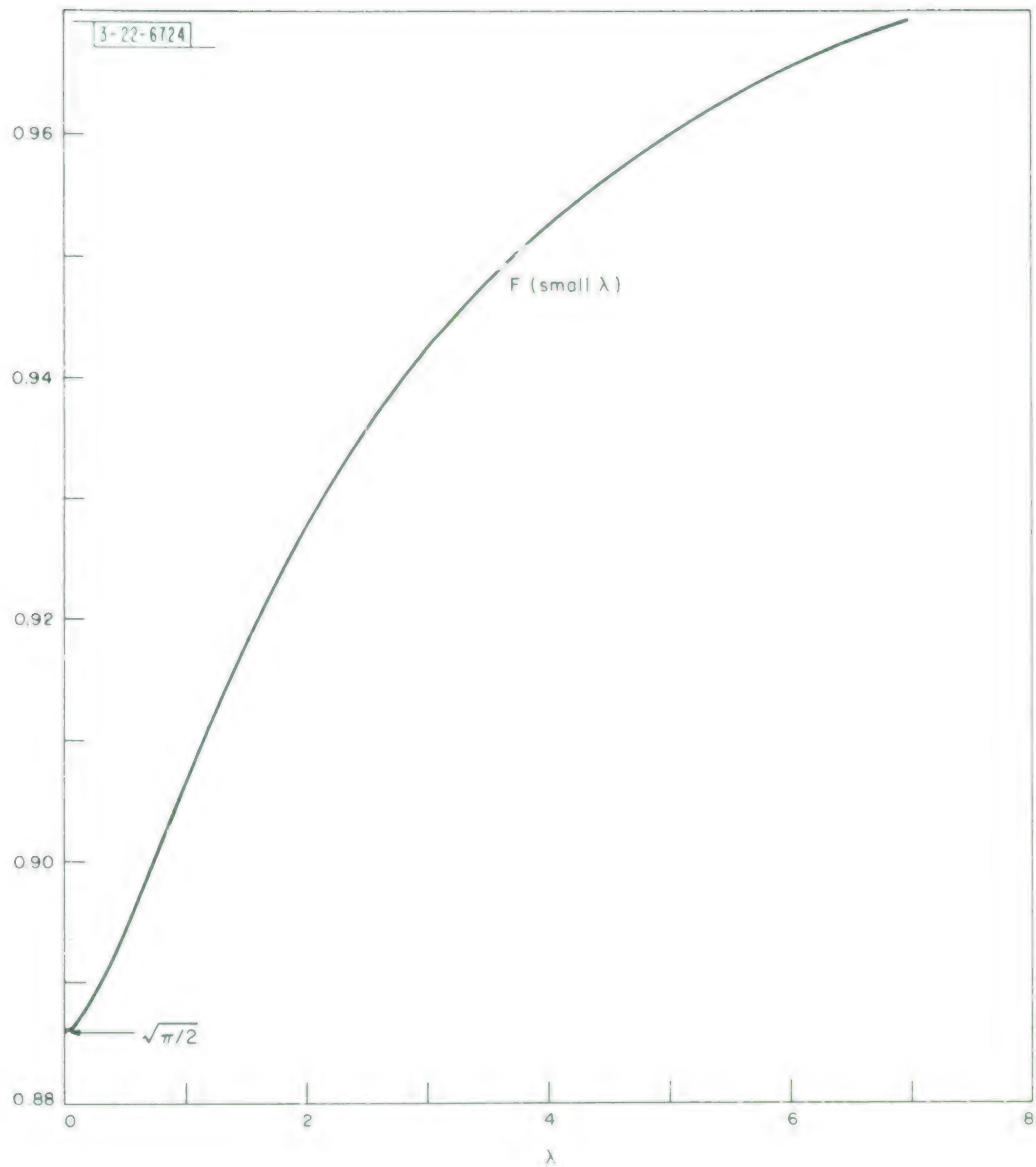


Fig. 1a

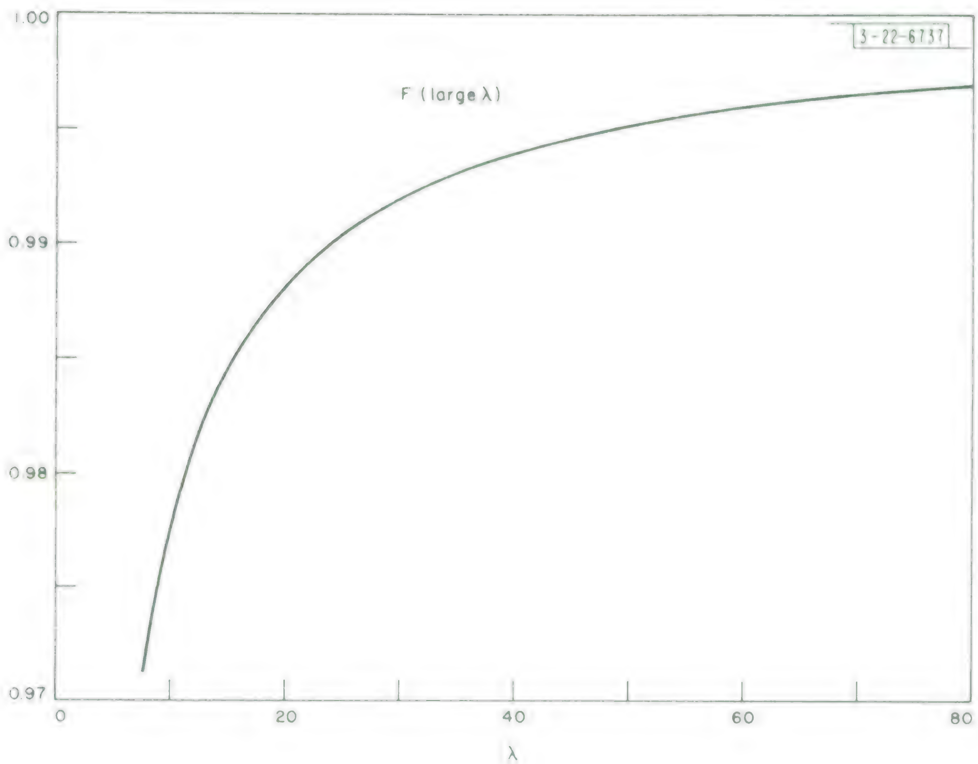


Fig. 1b

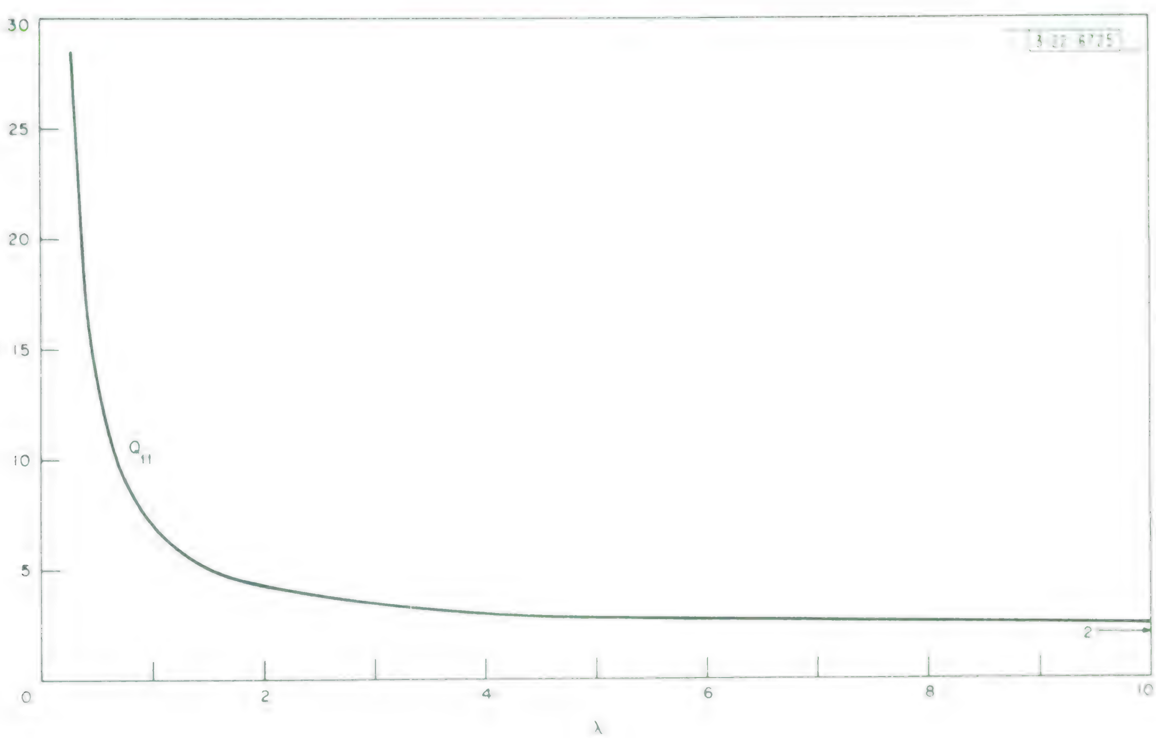


Fig. 2

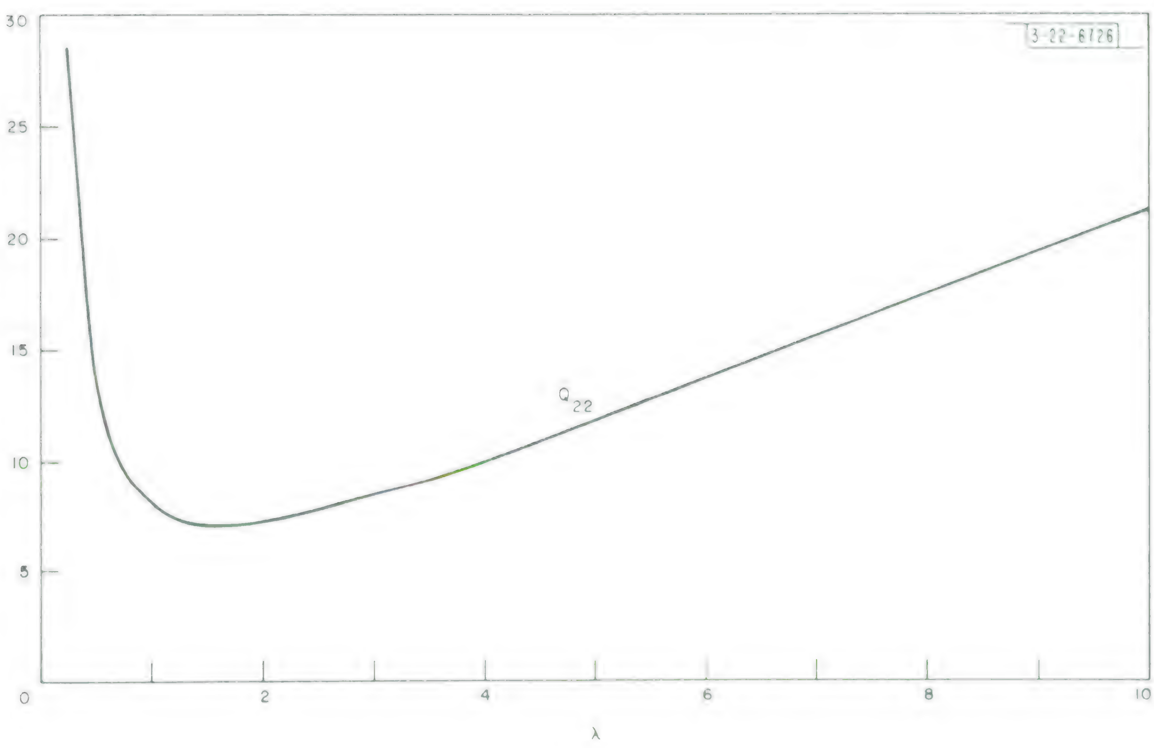


Fig. 3

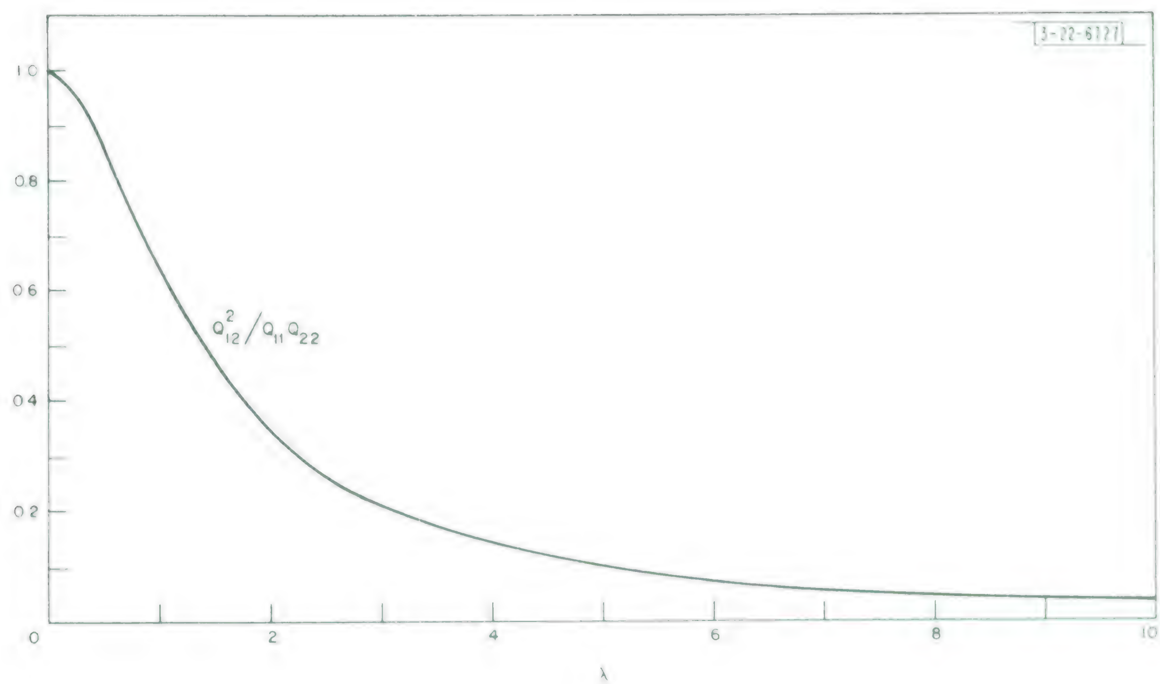
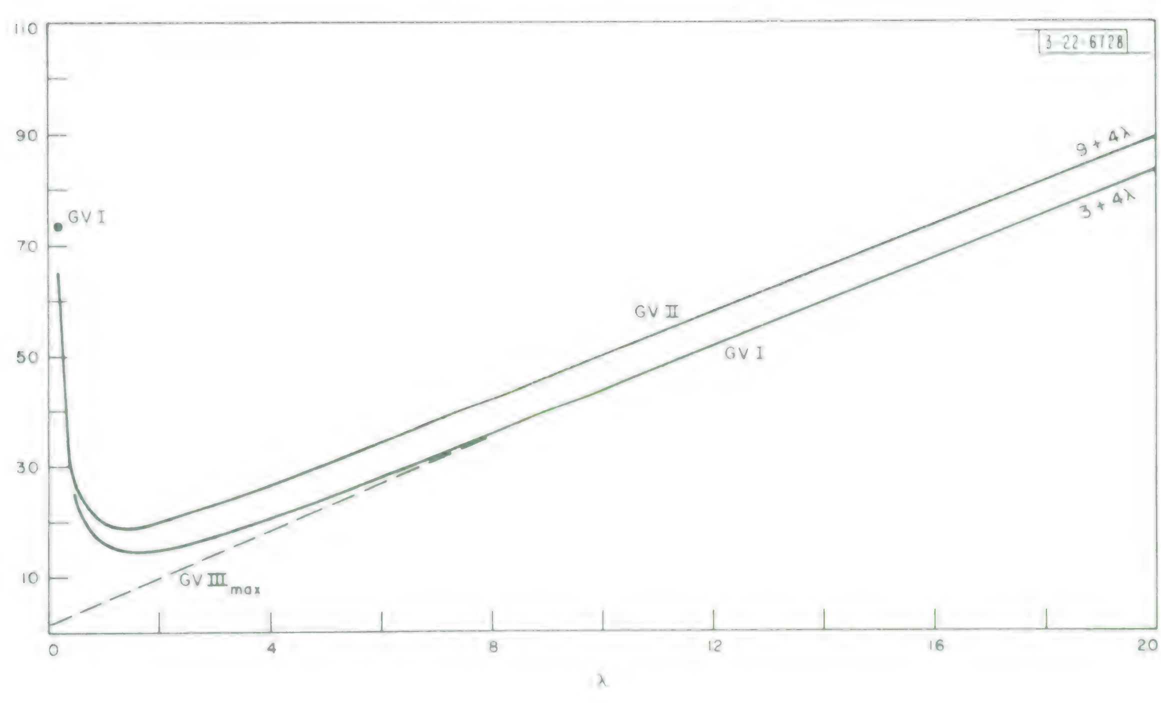


Fig. 4



3-22-6728
110
90
70
50
30
10
0
4
8
12
16
20
λ
GV I
GV II
GV I
GV III max
9 + 4λ
3 + 4λ

Fig. 5

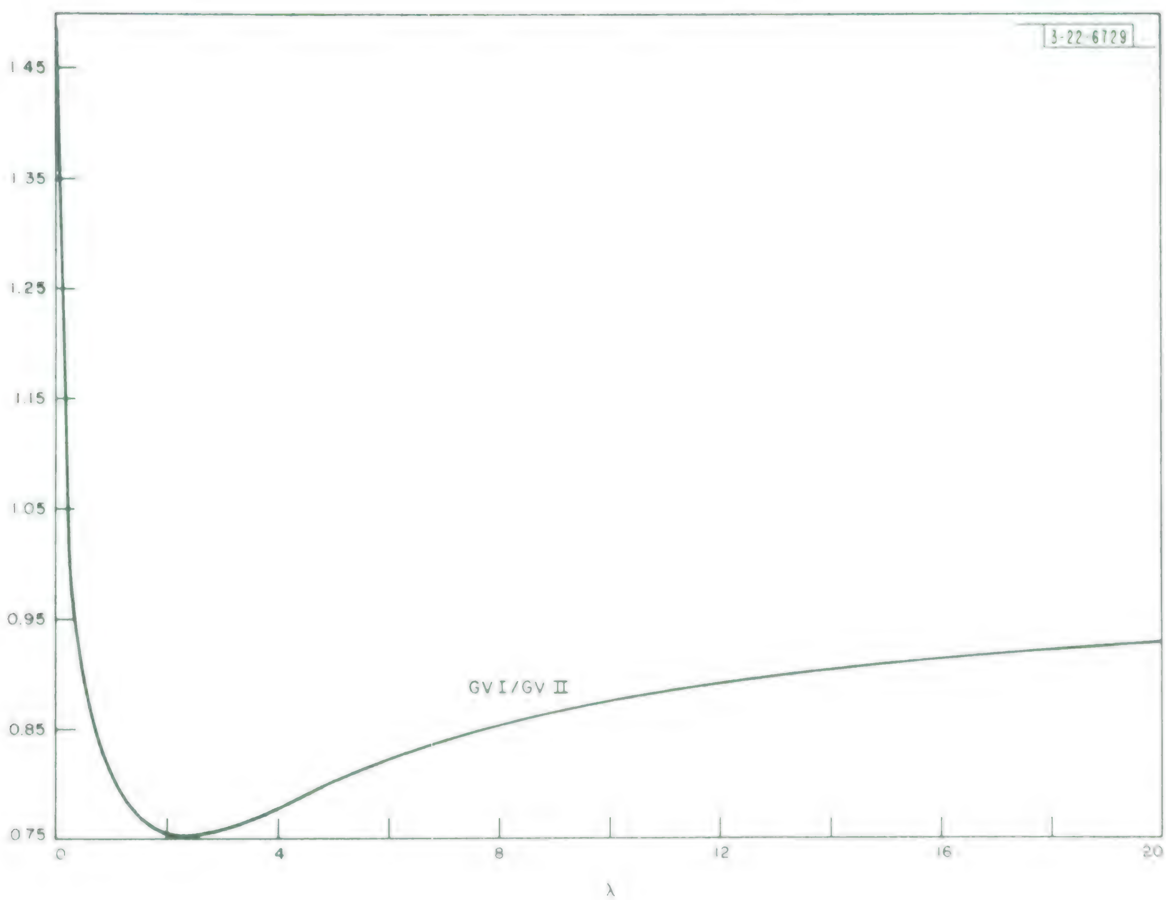


Fig. 6

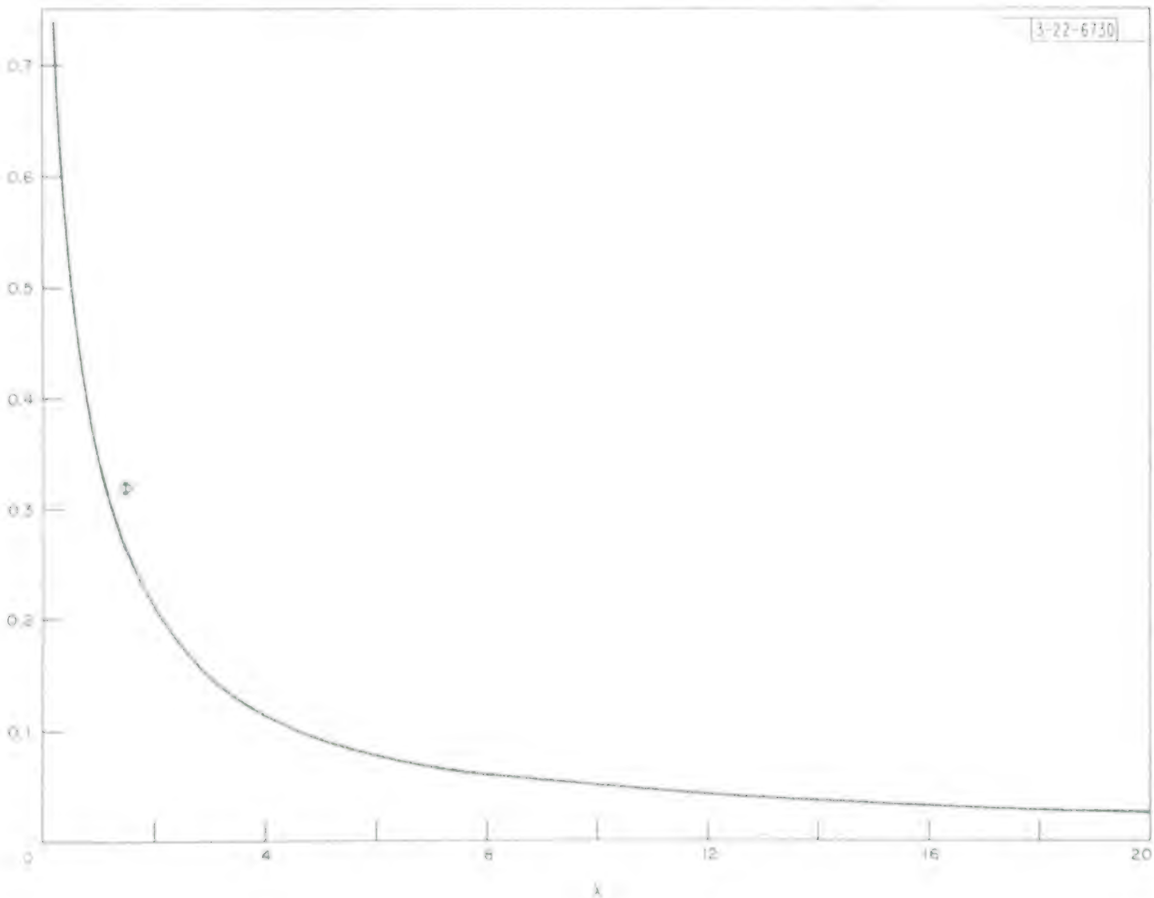


Fig. 7

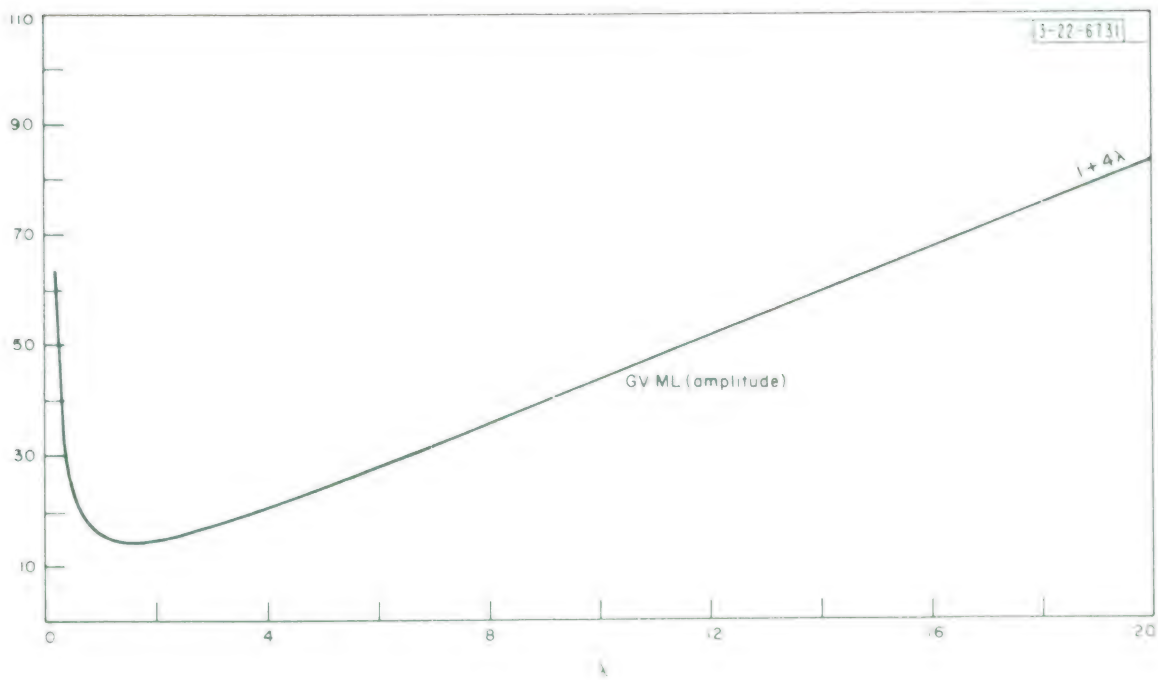


Fig. 8

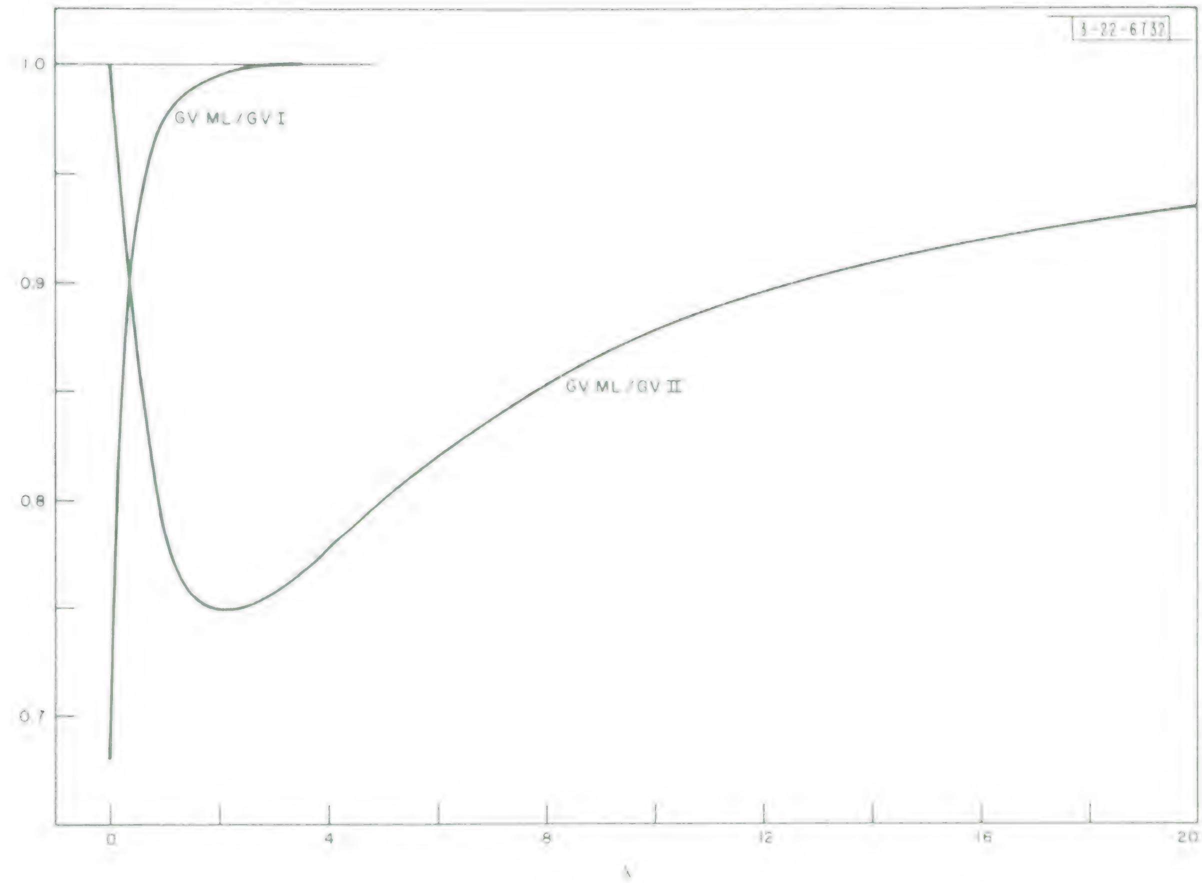


Fig. 9

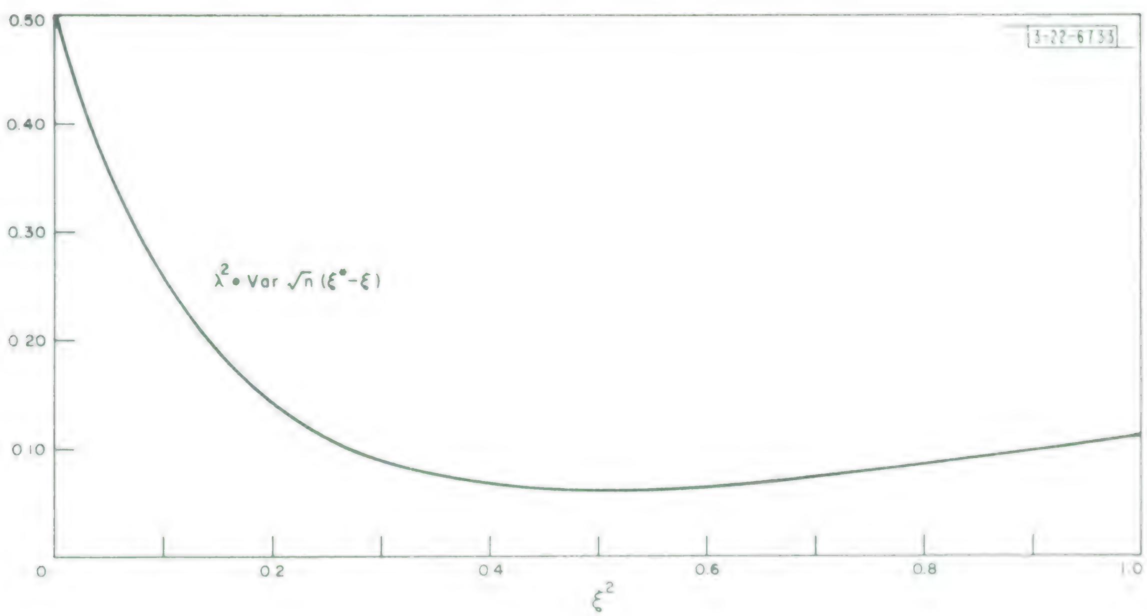

3-22-6733
0.50
0.40
0.30
0.20
0.10
0
0.2
0.4
0.6
0.8
1.0
$\lambda^2 \cdot \mathrm{Var}\,\sqrt{n}\,(\xi^* - \xi)$
ξ^2

Fig. 10

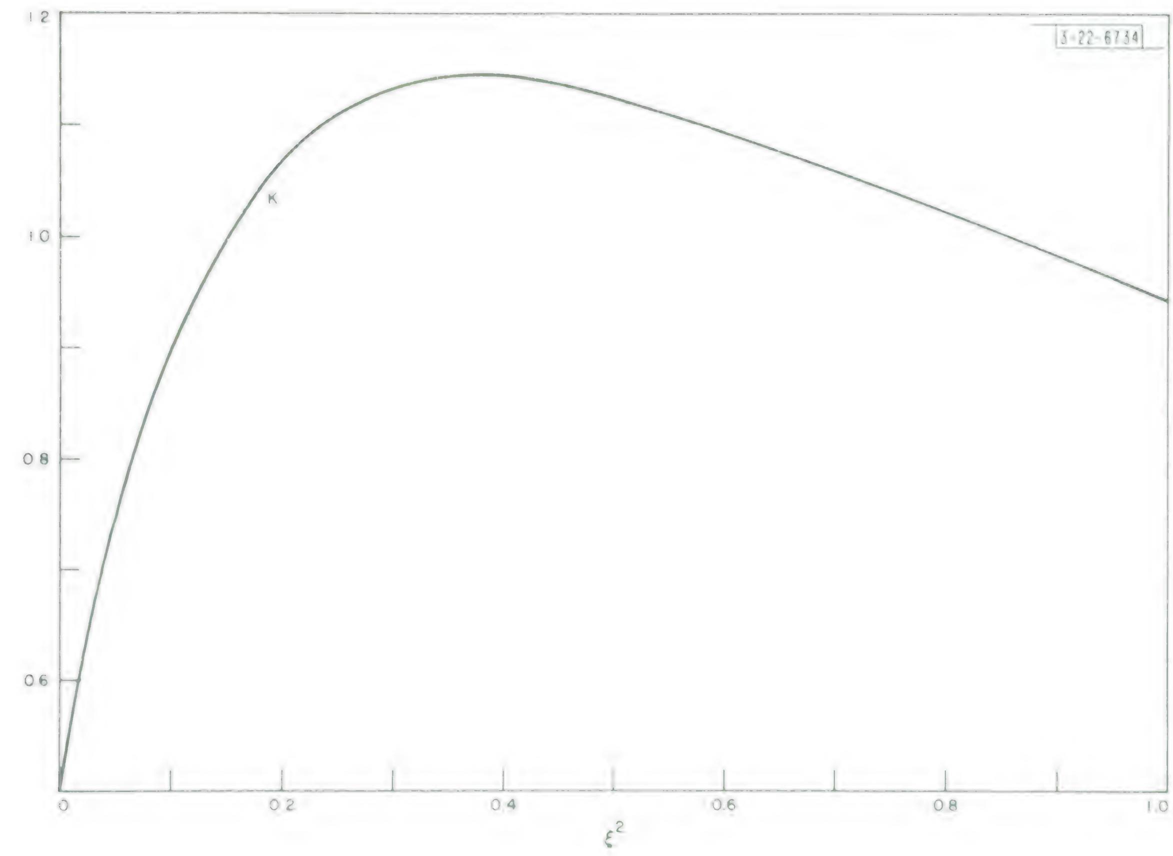
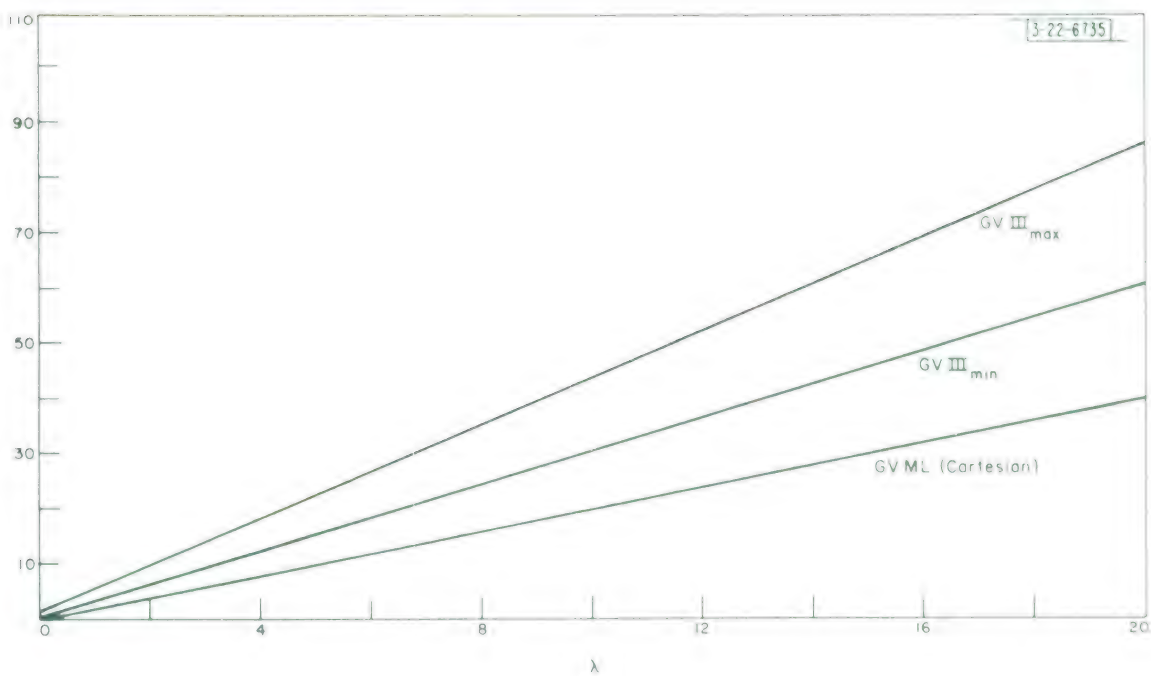


Fig. 11



110
90
70
50
30
10
0
4
8
12
16
20
λ
3-22-6735
GV III max
GV III min
GV ML (Cartesian)

Fig. 12

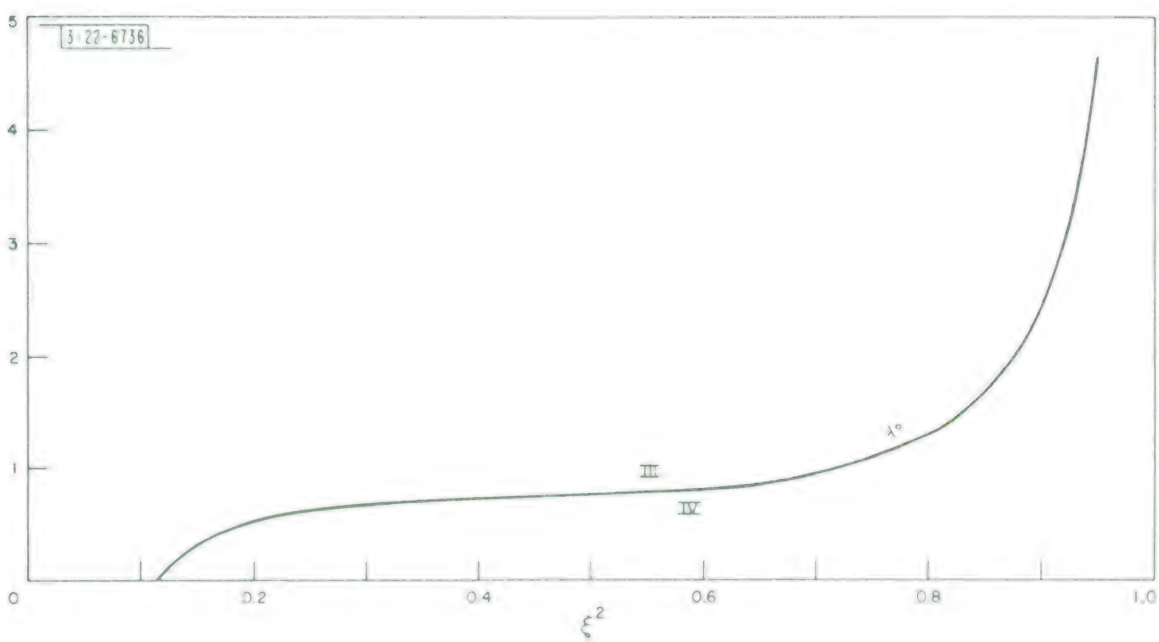


Fig. 13

REFERENCES

[1] Handbook of Mathematical Functions. National Bureau of Standards Applied Mathematics Series No. 55.

[2] Cramér, H., Mathematical Methods of Statistics (Princeton University Press, N.J., 1946).

[3] Gardner, L. A., Jr., "Rapid Frequency Estimation in the Presence of Unknown Phase and Amplitude," JA-2603, Lincoln Laboratory, M.I.T. Submitted to Technometrics.

[4] Kendall, M.G. and Stuart, A., The Advanced Theory of Statistics, Vol. 2 (Hafner, New York, 1961).

[5] Knopp, K., Theory and Application of Infinite Series (Hafner, New York, 1947).

[6] Parzen, E., Modern Probability Theory and its Applications (Wiley, New York, 1960).

[7] Wax, N., Selected Papers on Noise and Stochastic Processes (Dover, New York, Ed. 1954).

ADDENDUM

ESTIMATION OF DOPPLER FREQUENCY AND ACCELERATION USING PHASE SINES AND COSINES

Our concern has been joint estimation of the component powers (1.3). Methods I and II did this using only amplitude data, which is distributed independently of the phase φ_t in (1.1). Methods III and IV assumed (1.2) to be the case and utilized both amplitudes and phases (after conversion to Cartesian coordinates).

Here we want to estimate phase parameters from the data $p_1, p_2, \ldots, p_n$. The density of p_t is $f(p-\varphi_t)$ for $0 \le p \le 2\pi$ (or any interval of length 2π), and is otherwise zero. The function f is even about the origin and depends only on $\rho/\sigma > 0$. The expected value of $p_t - \varphi_t$ is not zero, but that of any odd 2π-periodic function of this difference is. As a result, we are able to construct consistent estimates of the coefficients in any polynomial phase model which use only the sines and cosines of the phase data. Such estimates are desirable because of the ambiguity in the phase data. If this were resolved, then the estimates become considerably simpler to compute.

We consider only the simplest generalization of (1.2); viz.,

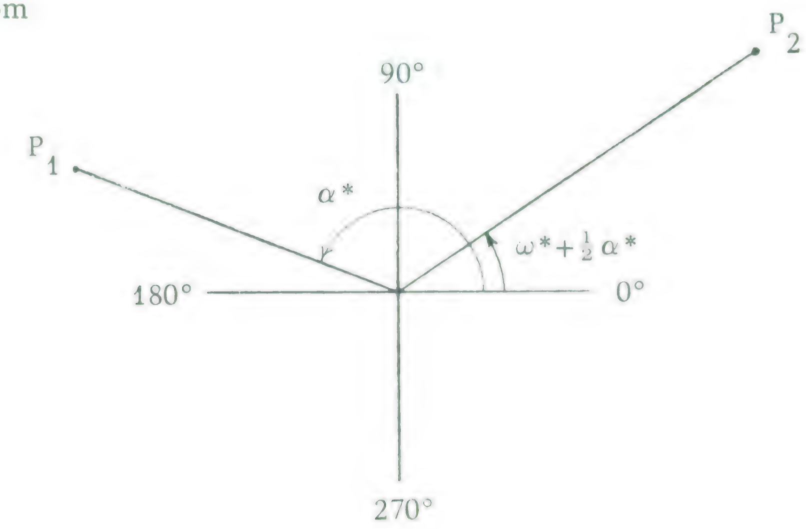
$$\varphi_t = \varphi + \omega t + \alpha \frac{t^2}{2} \tag{AD. 1}$$

where ω, as before, is the Doppler frequency and α is the new acceleration parameter. The estimate (ω^*, α^*) defined in the next paragraph converges to (ω, α) with probability one as $n \to \infty$, no matter what the unknown value of $\rho/\sigma > 0$.

Let the cosines and sines of the independent random variables $p_1, p_2, \ldots, p_n$ be given. Compute the four averages

$$\begin{aligned} X_1 &= \frac{1}{n} \sum_{t=1}^{n-2} \cos \Delta^2 p_t & Y_1 &= \frac{1}{n} \sum_{t=1}^{n-2} \sin \Delta^2 p_t \\ X_2 &= \frac{1}{n} \sum_{t=1}^{n-4} \cos(\Delta p_t - t\, \Delta^2 p_{t+2}) & Y_2 &= \frac{1}{n} \sum_{t=1}^{n-4} \sin(\Delta p_t - t\Delta^2 p_{t+2}) \end{aligned} \tag{AD. 2}$$

wherein Δ denotes the first forward difference operator. Graphically, the estimates are obtained from



where $P_i = (X_i, Y_i)$. If necessary, 360° is to be added to the difference to put ω^* between 0° and 360°. The limiting amplitudes are rather complicated homogeneous functions of the noncentral parameter $\lambda = \rho^2/2\sigma^2$.

If α is known, we estimate ω using

$$X_0 = \frac{1}{n} \sum_{t=1}^{n-1} \cos[\Delta p_t - \alpha(t+\tfrac{1}{2})] \qquad Y_0 = \frac{1}{n} \sum_{t=1}^{n-1} \sin[\Delta p_t - \alpha(t+\tfrac{1}{2})] \quad . \tag{AD. 3}$$

The point (X_0, Y_0) converges to $(A \cos \omega,\ A \sin \omega)$ where $A = A(\lambda)$.

The computation of P_0 and P_1 in terms of the $\cos p_t$'s and $\sin p_t$'s is a simple task relative to computing P_2. For the last, we must use the fact that

$$\cos kp = T_k(\cos p)$$

$$\sin kp = \sin p \; U_{k-1}(\cos p),$$

where T_k and U_k are the first and second kind Tchebichev polynomials of degree k.

A certain amount of recursive calculation is possible.

Phase density and trigonometric moments. We first derive the density of p_t with φ_t left arbitrary. As in Sec. 3 we take the bivariate circular Gaussian distribution, centered on the point $(\rho \cos \varphi_t, \rho \sin \varphi_t)$ with variance σ^2, and make the change of variables to polar coordinates. After appropriately completing the square, the probability density function of p_t becomes

$$f_t(p) = \frac{1}{2\pi} e^{-\frac{1}{2} a^2 \sin^2(p-\varphi_t)} \int_0^\infty z e^{-\frac{1}{2}[z - a\cos(p-\varphi_t)]^2} dz \, .$$

Here we have used

$$a = \rho/\sigma$$

for the Rice parameter to avoid confusion with the acceleration (cf. B.1). After adding in and subtracting out $a\cos(p-\varphi_t)$ from the integrand, and then integrating by parts, there results

$$f_t(p) = f(p-\varphi_t) \qquad (0 \le p \le 2\pi)$$

$$f(\theta) = \frac{1}{2\pi}\left[e^{-\frac{1}{2}a^2} + a\cos\theta \, e^{-\frac{1}{2}a^2 \sin^2\theta} \int_{-\infty}^{a\cos\theta} e^{-\frac{1}{2}t^2} dt \right]. \qquad \text{(AD. 4)}$$

When a, i.e. ρ, vanishes this properly reduces to a uniform distribution over $(0, 2\pi)$.

From (AD. 4) it is clear that the random variables

$$u_t = \cos(p_t - \varphi_t) \qquad v_t = \sin(p_t - \varphi_t) \qquad \text{(AD. 5)}$$

have the same joint distribution for all t. We will need their first and second moments in order to compute the limiting values of the estimates.

Letting u and v denote the common variates we have

$$\mathcal{E}u = \int_0^{2\pi} \cos(p-\varphi_t) f_t(p)dp = \int_{-\varphi_t}^{2\pi-\varphi_t} \cos\theta\, f(\theta)d\theta = \int_0^{2\pi} \cos\theta\, f(\theta)d\theta$$

because the integrand is periodic in θ with period 2π. Since $\cos\theta$ times the leading term in (AD. 4) integrates to zero,

$$\mathcal{E}u = \frac{a}{2\pi}\int_0^{2\pi} \cos^2\theta\, e^{-\frac{1}{2}a^2\sin^2\theta}\, G(a\cos\theta)d\theta$$

wherein

$$G(x) = \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$$

is $\sqrt{2\pi}$ times the standardized normal distribution function. We split $\mathcal{E}u$ into the integral from 0 to π plus π to 2π, and replace θ by $\theta+\pi$ in the latter. This changes the sign of $\cos\theta$ and $\sin\theta$:

$$\int_\pi^{2\pi} = \int_0^{\pi} \cos^2\theta\, e^{-\frac{1}{2}a^2\sin^2\theta}\, G(-a\cos\theta)d\theta.$$

But

$$G(-x) = \sqrt{2\pi} - G(x)$$

for all x, so

$$\mathcal{E}u = \frac{a}{\sqrt{2\pi}} \int_0^{\pi} \cos^2\theta\, e^{-\frac{1}{2}a^2\sin^2\theta}\, d\theta$$

$$= \frac{a}{\sqrt{2\pi}} \left(1 + \frac{d}{d\lambda}\right) \int_0^{\pi} e^{-\lambda\sin^2\theta}\, d\theta$$

where

$$\lambda = a^2/2$$

is the noncentral parameter. Substituting $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and then $\theta = \varphi/2$ gives (see (3.2))

$$\int_0^{\pi} e^{-\lambda\sin^2\theta} = \frac{1}{2} e^{-\frac{\lambda}{2}} \int_0^{2\pi} e^{\frac{\lambda}{2}\cos\varphi} d\varphi = \pi e^{-\frac{\lambda}{2}} I_0(\frac{\lambda}{2})$$

Since the derivative of I_0 is I_1, the preceding combine to give

$$\mathcal{E}u = \frac{\sqrt{\pi\lambda}}{2} e^{-\frac{\lambda}{2}} [I_0(\frac{\lambda}{2}) + I_1(\frac{\lambda}{2})] \tag{AD.6}$$

for the mean value of $\cos(p_t - \varphi_t)$.

We compute the mean value of $\sin^2(p_t - \varphi_t)$ by the following device. Define

$$J = \int_0^{2\pi} \sin^2\theta \cos\theta \, e^{-\frac{1}{2}a^2\sin^2\theta} G(a\cos\theta)d\theta$$

$$K = \int_0^{2\pi} \cos\theta \, e^{-\frac{1}{2}a^2\sin^2\theta} G(a\cos\theta)d\theta ,$$

which we view as functions of a. Then

$$\mathcal{E}v^2 = \int_0^{2\pi} \sin^2\theta \, f(\theta)d\theta = \frac{1}{2} e^{-\frac{1}{2}a^2} + \frac{a}{2\pi} J$$

$$1 = \int_0^{2\pi} f(\theta)d\theta = e^{-\frac{1}{2}a^2} + \frac{a^2}{2\pi} K .$$

If we differentiate the integral representation for K we get

$$\frac{dK}{da} = \pi e^{-\frac{1}{2}a^2} - aJ$$

$$= 2\pi\left[e^{-\frac{1}{2}a^2} - \mathcal{E}v^2\right] .$$

The derivative of the other expression is

$$\frac{dK}{da} = 2\pi\left[e^{-\frac{1}{2}a^2} - \frac{1}{a^2}(1-e^{-\frac{1}{2}a^2})\right] .$$

Thus

$$\mathcal{E}v^2 = \frac{1-e^{-\lambda}}{2\lambda} = 1-\mathcal{E}u^2 \qquad \text{(AD. 7)}$$

after equating the two formulae.

Any odd power of v is orthogonal to any function of u, and hence they are uncorrelated. We have, for $k=0, 1, 2, \ldots,$

$$\mathcal{E}v^{2k+1}g(u) = \frac{1}{2}\int_{-2\pi}^{2\pi} \sin\theta[\sin^{2k}\theta\, g(\cos\theta) f(\theta)]\, d\theta$$

because the integrand has period 2π. Since the function within square brackets is even the integrand is an odd function, so

$$\mathcal{E}v^{2k+1}g(u) = 0. \qquad \text{(AD. 8)}$$

In particular,

$$\mathcal{E}v = 0 \qquad \mathcal{E}uv = 0. \qquad \text{(AD. 9)}$$

(u, v) is an example of a pair of uncorrelated, yet obviously dependent, random variables.

Estimation of frequency when acceleration is known. The method for analyzing the large sample behavior of the points P_i $(i=0,1,2)$ is the same in each case, but gets more complicated with increasing i. We will therefore start with the averages (AD. 3). We have

$$\begin{aligned} \Delta p_t &= \omega + \alpha(t+\tfrac{1}{2}) + \Delta(p_t - \varphi_t) \\ \Delta^2 p_t &= \alpha + \Delta^2(p_t - \varphi_t) , \end{aligned} \tag{AD. 10}$$

only the first of these being needed at the present. Setting

$$X'_0 = \frac{1}{n}\sum_{t=1}^{n-1} \cos\Delta(p_t - \varphi_t) \qquad Y'_0 = \frac{1}{n}\sum_{t=1}^{n-1} \sin\Delta(p_t - \varphi_t) ,$$

we have

$$\begin{aligned} X_0 &= X'_0 \cos\omega - Y'_0 \sin\omega \\ Y_0 &= X'_0 \sin\omega + Y'_0 \cos\omega . \end{aligned} \tag{AD. 11}$$

In the notation (AD. 5)

$$\begin{aligned} \cos\Delta(p_t - \varphi_t) &= u_t u_{t+1} + v_t v_{t+1} \\ \sin\Delta(p_t - \varphi_t) &= u_t v_{t+1} - v_t u_{t+1} . \end{aligned} \tag{AD. 12}$$

Since (u_t, v_t) are independent and identically distributed we have, using (AD. 6) and (AD. 9)

$$\mathcal{E} \cos\Delta(p_t - \varphi_t) = A$$

$$\mathcal{E} \sin\Delta(p_t - \varphi_t) = 0 .$$

We have set

$$A = \mathcal{E}\overset{2}{u} = \frac{\pi\lambda}{4}e^{-\lambda}\left[I_0(\tfrac{\lambda}{2}) + I_1(\tfrac{\lambda}{2})\right]^2$$

$$B = \mathcal{E}u^2 = 1 - \frac{1-e^{-\lambda}}{2\lambda} \; ; \tag{AD. 13}$$

the latter will be used subsequently. If we subtract the constant A from X_0' we obtain the arithmetic mean of first-order dependent, zero mean, bounded random variables. The same is true of Y_0' as it stands. A trivial application of the Law of Large Numbers for dependent random variables (used in Appendix C) shows that (X_0', Y_0') converges to the point $(A, 0)$ as $n \to \infty$ with probability one and in mean square. Thus, in both these senses,

$$X_0 \to A\cos\omega \qquad\qquad Y_0 \to A\sin\omega \; . \tag{AD. 14}$$

As claimed, the phase of the point P_0 consistently estimates the unknown ω. The function $A(\lambda)$ increases steadily from $A(0) = 0$ to $A(+\infty) = 1$. As we would expect, estimation is difficult when λ is small.

<u>The large sample distribution of P_0</u>. For the second moments of the random variables in (AD. 12) we find, in the notation (AD. 13),

$$\mathcal{E}\cos\Delta(p_t-\varphi_t)\cos\Delta(p_s-\varphi_s) = \begin{cases} B^2 + (1-B)^2 & \text{for } s=t \\ AB & \text{for } |s-t| = 1 \\ A^2 & \text{for } |s-t| \geq 2 \end{cases}$$

$$\mathcal{E}\sin\Delta(p_t-\varphi_t)\sin\Delta(p_s-\varphi_s) = \begin{cases} 2B(1-B) & \text{for } s=t \\ -A(1-B) & \text{for } |s-t| = 1 \\ 0 & \text{for } |s-t| \geq 2 \end{cases}$$

$$\mathcal{E}\cos\Delta(p_t-\varphi_t)\sin\Delta(p_s-\varphi_s) = 0 \qquad \text{for all } t, s.$$

We have, after replacing n-1 by n in the upper limit of the sum which is X'_0,

$$n\mathcal{E}(X'_0 - A)^2 = \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}\mathcal{E}[\cos\Delta(p_t-\varphi_t)-A]\,[\cos\Delta(p_s-\varphi_s)-A]$$

$$= \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}(\sigma_{t-s} - A^2).$$

We have set $\sigma_{t-s} = \mathcal{E}\cos\Delta(p_t-\varphi_t)\cos\Delta(p_s-\varphi_s) = \sigma_{s-t}$. There are $n-|k|$ indices $t, s = 1, 2, \ldots, n$ for which $t-s = k$ $(k=0, \pm1, \ldots, \pm(n-1))$. Hence

$$n\mathcal{E}(X'_0 - A)^2 = \sum_{|k| \leq n-1}\left(1 - \frac{|k|}{n}\right)(\sigma_k - A^2)$$

$$= (\sigma_0 - A^2) + 2(1 - \frac{1}{n})(\sigma_1 - A^2),$$

and

$$\lim_n n\mathcal{E}(X'_0 - A)^2 = B^2 + (1-B)^2 - A^2 + 2(AB - A^2).$$

Similarly, we find

$$\lim_n n\mathcal{E}Y_0'^2 = 2(1-B)(B-A)$$

Furthermore

$$\mathcal{E}(X'_0 - A)\,Y'_0 = 0$$

for every n.

By Liapounov's Central Limit Theorem it follows, for $n \to \infty$, that $\sqrt{n}(X'_0 - A)$ and $\sqrt{n}\,Y'_0$ tend to bivariate normality about (0, 0) with covariance matrix given by

the above limiting values of the corresponding second moments. According to (AD. 11), $\sqrt{n}(X_0 - A\cos\omega)$ and $\sqrt{n}(Y_0 - A\sin\omega)$ are obtained from these variates by a rotation through the angle ω. Thus, introducing

$$C_1 = (1-B)^2 + (B-A)(B+3A)$$

$$C_2 = 2(B-A)(1-B),$$

we see that $\sqrt{n}(X_0 - A\cos\omega)$ and $\sqrt{n}(Y_0 - A\sin\omega)$ have a large sample normal distribution about the origin with covariance matrix

$$\begin{bmatrix} C_1\cos^2\omega + C_2\sin^2\omega & (C_1-C_2)\sin\omega\cos\omega \\ (C_1-C_2)\sin\omega\cos\omega & C_1\sin^2\omega + C_2\cos^2\omega \end{bmatrix} \qquad \text{(AD. 15)}$$

From (AD. 13) we have $0 < A < B < 1$, so the C's are positive functions of λ.

<u>Joint estimation of frequency and acceleration.</u> We now consider the point P_1 in (AD. 2). If we set

$$X_1' = \frac{1}{n}\sum_{t=1}^{n-2}\cos\Delta^2(p_t - \varphi_t) \qquad Y_1' = \frac{1}{n}\sum_{t=1}^{n-2}\sin\Delta^2(p_t - \varphi_t),$$

then, according to (AD. 10), (AD. 11) is true with ω replaced by α and the 0 subscripts replaced by 1's. The expression comparable with (AD. 12) is

$$\begin{aligned} \cos\Delta^2(p_t-\varphi_t) = {} & u_t(2u_{t+1}^2-1)u_{t+2} + 2v_tu_{t+1}v_{t+1}u_{t+2} \\ & -v_t(2u_{t+1}^2-1)v_{t+2} + 2u_tu_{t+1}v_{t+1}v_{t+2} \\ \sin\Delta^2(p_t-\varphi_t) = {} & v_t(2u_{t+1}^2-1)u_{t+2} - 2u_tu_{t+1}v_{t+1}v_{t+2} \\ & + u_t(2u_{t+1}^2-1)v_{t+1} + 2v_tu_{t+1}v_{t+1}v_{t+2} . \end{aligned} \qquad \text{(AD. 16)}$$

From these and (AD. 6) - (AD.8) we find

$$\mathcal{E}\cos\Delta^2(p_t-\varphi_t) = \mathcal{E}^2 u(2\mathcal{E}u^2-1) = A(2B-1)$$
$$\mathcal{E}\sin\Delta^2(p_t-\varphi_t) = 0 \tag{AD. 17}$$

Thus, for the same reasons and in the same senses,

$$X_1 \to A(2B-1)\cos\alpha \qquad Y_1 \to A(2B-1)\sin\alpha . \tag{AD. 18}$$

The function $B(\lambda)$ in (AD. 13) increases steadily from $B(0) = 1/2$ to $B(+\infty) = 1$, so the limiting amplitude is positive for all positive λ.

Turning now to the second point in (AD. 2), we have from (AD. 10)

$$\Delta p_t - t\Delta^2 p_t = \omega + \tfrac{1}{2}\alpha + \Delta(p_t - \varphi_t) - t\Delta^2(p_{t+2} - \varphi_{t+2})$$
$$= \omega + \tfrac{1}{2}\alpha + z_t \qquad \text{(say)}.$$

Again, (AD. 11) holds with ω replaced by $\omega+\frac{1}{2}\alpha$ and the 0 subscripts replaced by 2's when we set

$$X'_2 = \frac{1}{n}\sum_{t=1}^{n-4}\cos z_t \qquad Y'_2 = \frac{1}{n}\sum_{t=1}^{n-4}\sin z_t .$$

Since $\Delta(p_t - \varphi_t)$ and $\Delta^2(p_{t+2} - \varphi_{t+2})$ are independent random variables,

$$\mathcal{E}\cos z_t = A\,\mathcal{E}\cos t\Delta^2(p_{t+2} - \varphi_{t+2})$$
$$\mathcal{E}\sin z_t = A\,\mathcal{E}\sin t\,\Delta^2(p_{t+2} - \varphi_{t+2})$$

results from the display preceding (AD. 13). Let us now make the following redefinitions in the right side of (AD. 16) :

$$u_t = \cos t(p_{t+2} - \varphi_{t+2})$$

$$u_{t+1} = \cos t(p_{t+3} - \varphi_{t+3})$$

$$u_{t+2} = \cos t(p_{t+4} - \varphi_{t+4}) \quad , \tag{AD. 19}$$

and similarly for the v's with sines in place of the cosines. From the second display in (AD. 16) it is clear that the expectation of $\sin t\Delta^2(p_{t+2} - \varphi_{t+2})$ will vanish if that of $\sin t(p_{t+j} - \varphi_{t+j})$ does for all j. Such is the case:

$$\mathcal{E} \sin t(p_{t+j} - \varphi_{t+j}) = \int_{-\varphi_{t+j}}^{2\pi - \varphi_{t+j}} \sin t\theta\ f(\theta)d\theta = \int_0^{2\pi} \sin t\theta\, f(\theta)d\theta = 0$$

because t is an integer and the periodic integrand is odd. (If we had averaged the cosine and sine of $\Delta p_t - (t + \frac{1}{2})\Delta^2 p_t$ in an attempt to get ω directly, we would have had a $t + \frac{1}{2}$ in the integrand, making it a-periodic. $\mathcal{E} \sin z_t$ would therefore not be zero.) In the same way we have from the first display in (AD. 16)

$$\mathcal{E} \cos t\Delta^2(p_{t+2} - \varphi_{t+2}) = \mathcal{E} u_t (2\mathcal{E} u_{t+1}^2 - 1)\ \mathcal{E} u_{t+2} = A_t(2B_t - 1)$$

where the u's are defined in (AD. 19). $\mathcal{E} u_t$ and $\mathcal{E} u_{t+2}$ are equal, and we have set

$$A_t^{\frac{1}{2}} = \int_0^{2\pi} \cos t\theta\ f(\theta)d\theta \qquad B_t = \int_0^{2\pi} \cos^2 t\theta\, f(\theta)d\theta\ . \tag{AD. 20}$$

When t = 1, these become the $A^{\frac{1}{2}}$ and B of (AD. 13).

The integrals (AD. 20) are hard to evaluate. However, only the first need be considered because

$$2B_t - 1 = A_{2t}^{\frac{1}{2}}$$

for all t. (A_k is the value of the characteristic function at the point k of the random variable $p_t - \varphi_t$.) If we assume the sequence of the A's is such that the time average

$$\lim_{n} \frac{1}{n} \sum_{t=1}^{n} A_t A_{2t}^{\frac{1}{2}} = L \qquad \text{(AD. 21)}$$

exists (which is not an unreasonable supposition), then it is true that

$$X_2 \rightarrow AL\cos(\omega+\tfrac{1}{2}\alpha) \qquad Y_2 \rightarrow AL\sin(\omega+\tfrac{1}{2}\alpha) \qquad \text{(AD. 22)}$$

as $n \rightarrow \infty$.

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13. ABSTRACT

This report treats the theoretical problem of joint estimation of the wake and hard body cross sections (the parameters of the Rice distribution) from either incoherent or coherent radar data.

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